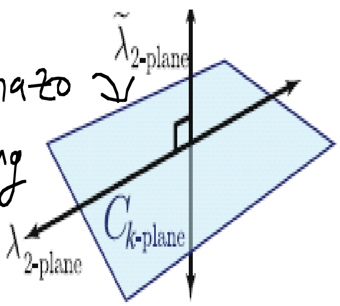
The Amplituhedron

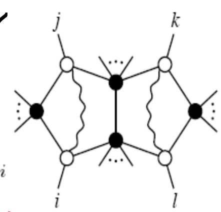
w/ Cachazo Cheung Kaplan



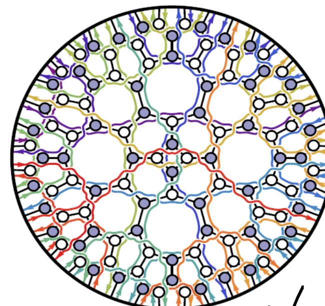
'09

w/ Bourjaily, Cachazo, Caron-Huot, Trnka + Hodges

$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$

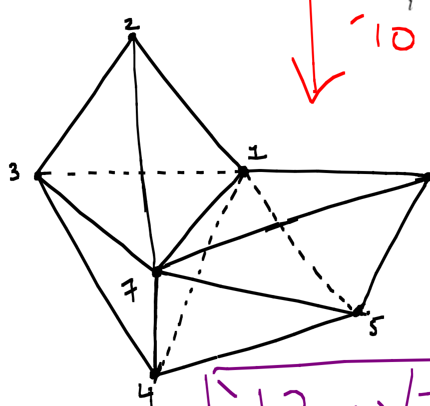


'10



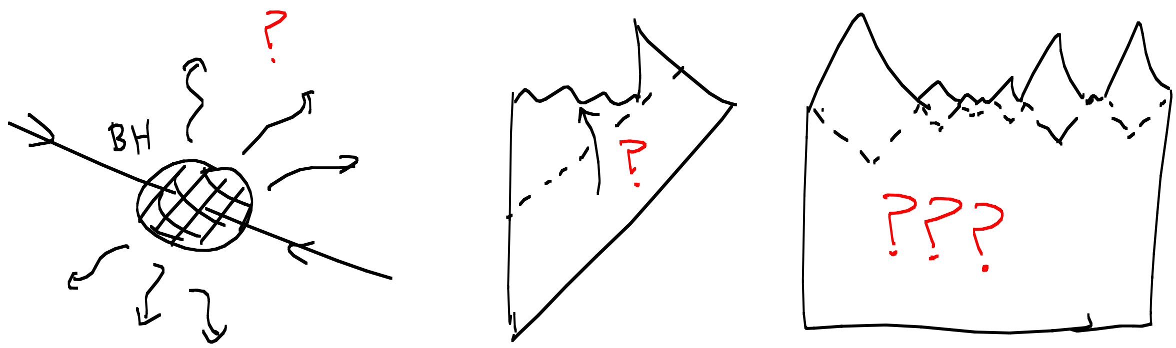
w/ Bourjaily Cachazo Goncharov Postnikov Trnka

'12



'13 w/ Trnka

Motivation



Emergent Space-Time

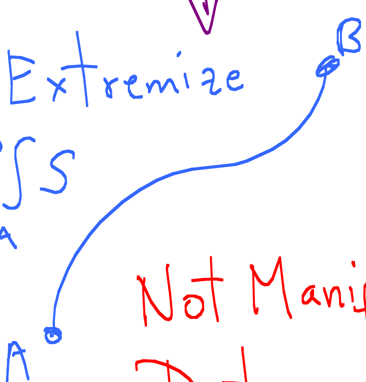
Emergent QM(?)

Classical $\longrightarrow$ Quantum

$F = ma$: Manifestly Deterministic

$\Big\downarrow$ Reformulate

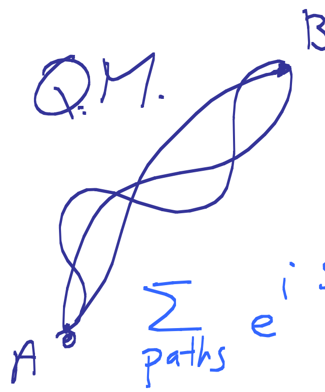
Extremize $\int_A^B S$



Not Manifestly Deterministic

$\xrightarrow{\text{Deform}}$

Q.M.



$\sum_{\text{paths}} e^{iS/\hbar}$

Strategy

1. Reformulate QFT, Eviscerating Locality + Unitarity → see them arise as emergent phenomena

2. Find Natural Deformation

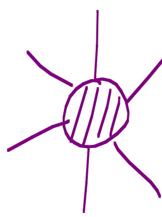
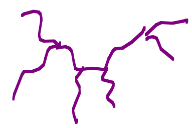
Step 1

0. Planar $\mathcal{N}=4$ SYM
1. Non-planar
2. Non-supersymmetric
3. Gravity
4. Perturbative Strings
5. AdS/CFT

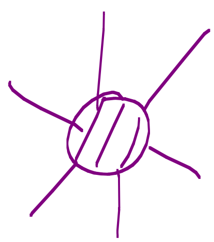
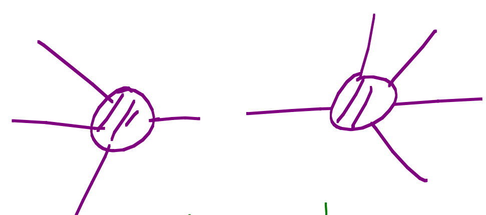
$\vdots$

Locality + Unitarity

* Obviously, hard-wired into usual QFT formalism

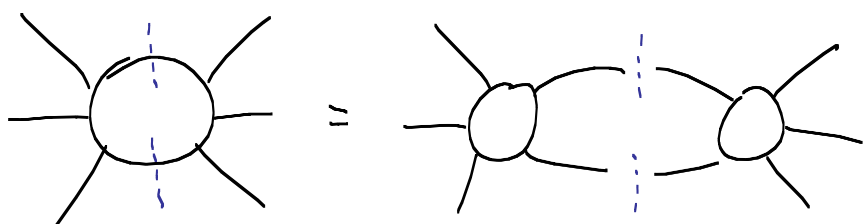
e.g. trees  $= \sum_i$  makes it obvious:

Locality : Poles where $\left(\sum_{L} p_i\right)^2 \to 0$

Unitarity : ∂  $=$ 

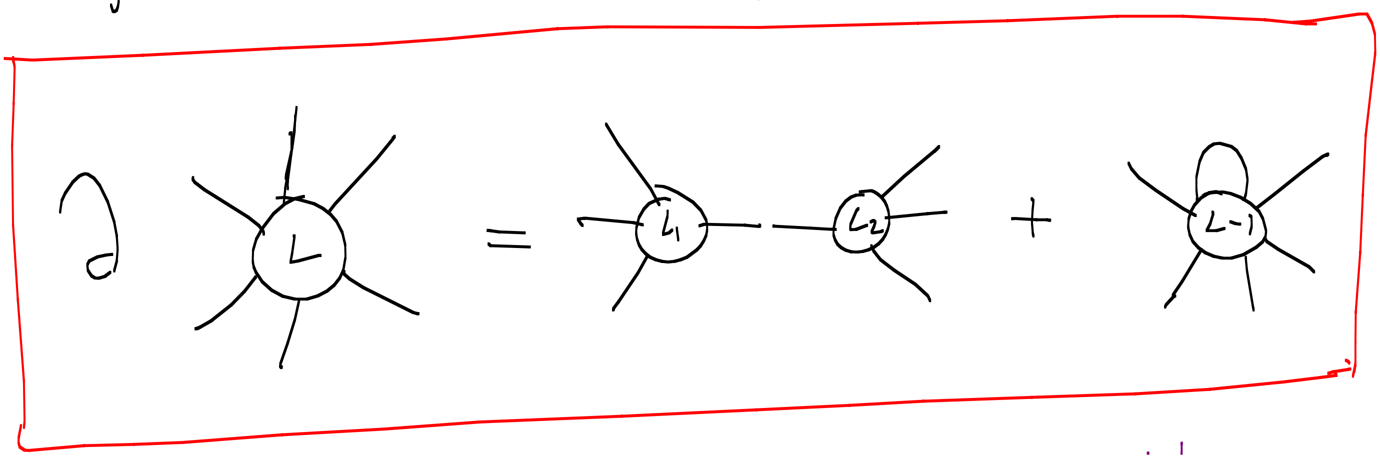
Factorization

L + U have an even simpler avatar in "cut"-structure of "integrand" for amplitudes

e.g. 

+ powerful "generalized" Unitarity

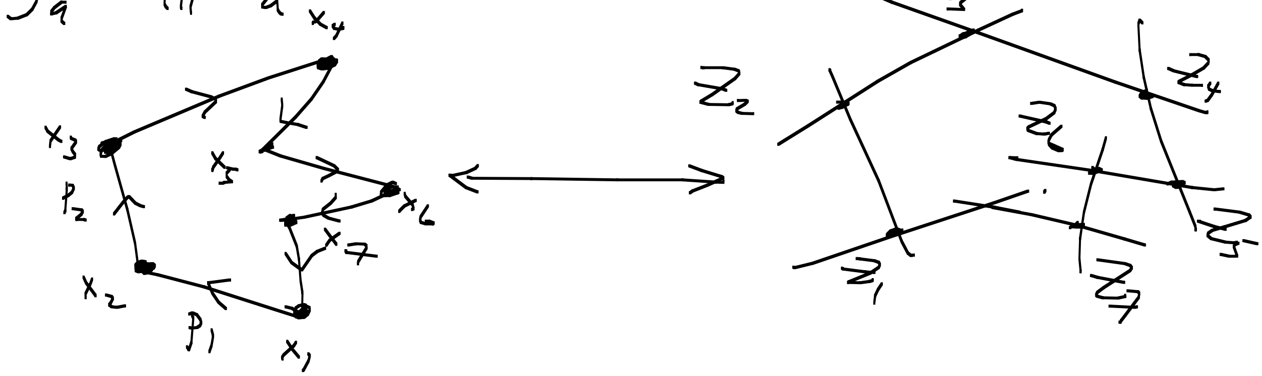
In planar $\mathcal{N} \geq 2$ theories,
most transparent + sharp statement
for THE integrand:



Planar $\mathcal{N}=4$ SYM

Kinematics I

$$p_a^\mu = x_{a+1}^\mu - x_a^\mu$$

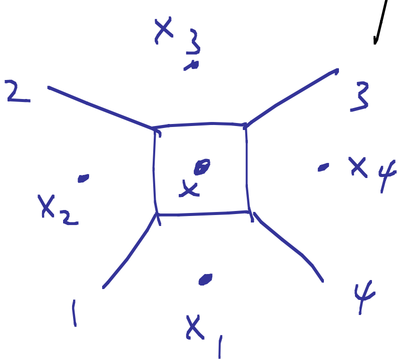


$$Z_a = \begin{pmatrix} \lambda_a \\ \mu_a \\ \hdashline \eta_a \end{pmatrix} \qquad Z_a \sim t_a Z_a$$ "momentum twistors"

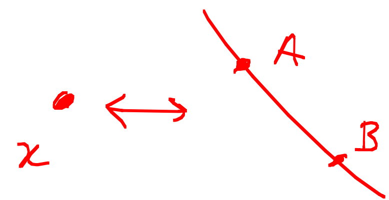
$$\tilde{\lambda}_a = \frac{\langle a-1\, a\rangle \mu_{a+1} + \langle a+1\, a-1\rangle \mu_a + \langle a\, a+1\rangle \mu_{a-1}}{\langle a-1\, a\rangle \langle a\, a+1\rangle}$$

Kinematics II

Planar Loop Integrand:



$$\frac{d^4 x}{(x-x_1)^2 \cdots (x-x_4)^2}$$

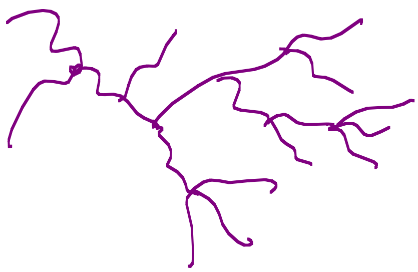


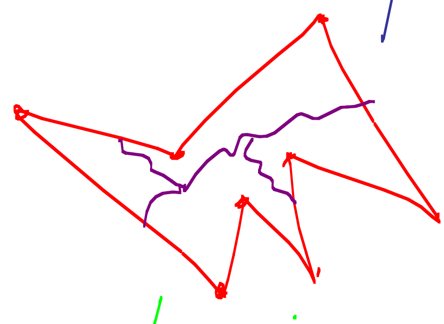
$$\mathcal{M}_{n,\hat{k},\ell} = \frac{\delta^4\left(\sum p_a\right)\delta^8\left(\sum_a \lambda_a \tilde{\eta}_a\right)}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

$$\times\, M_{n,k,\ell}\left[Z_a ; AB_1, AB_2, \ldots AB_\ell\right]$$

"Amplitude"

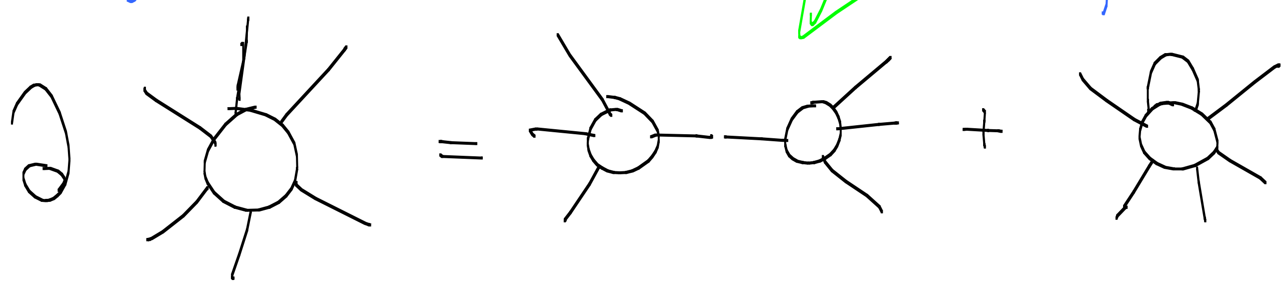
"Wilson-Loop"

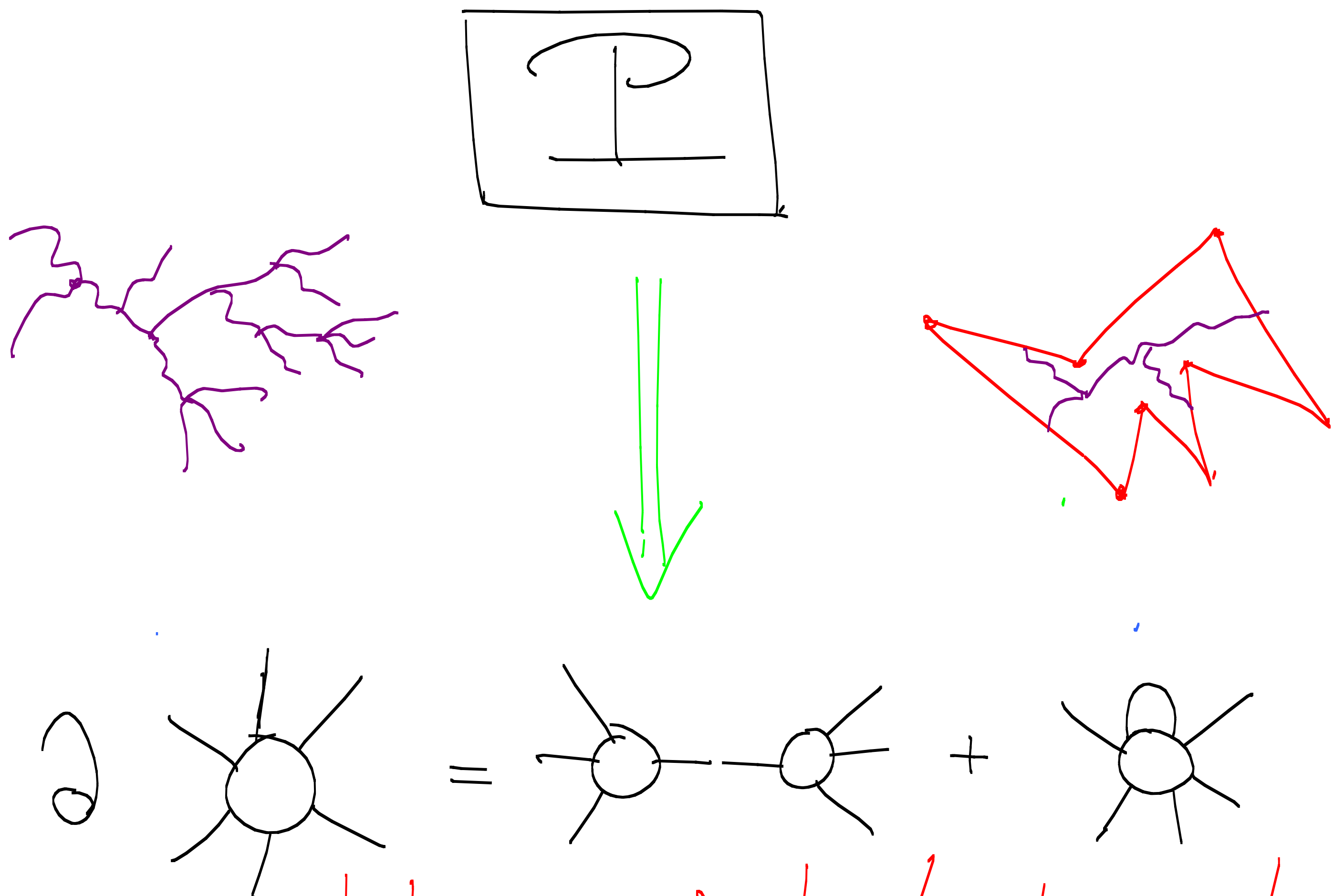




Conformal Manifest

Dual Conformal Manifest

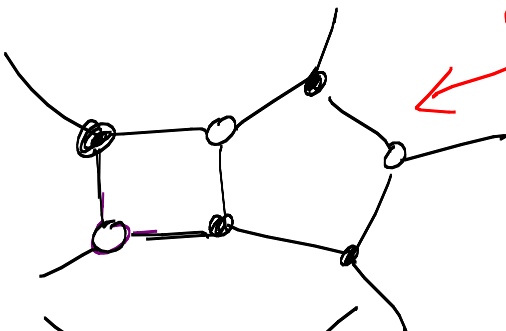


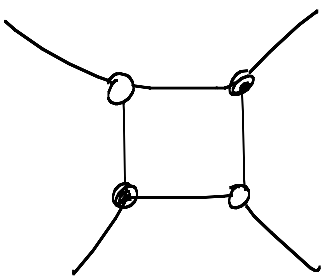


- Locality + Unitarity Emergent
- Infinite Yangian Symmetry Manifest

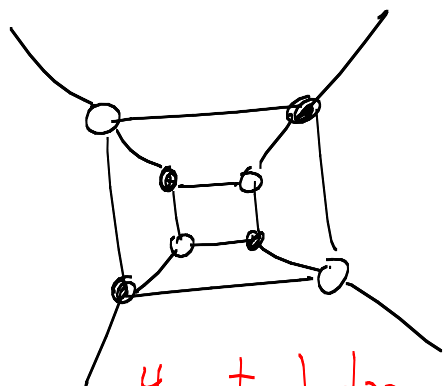
On-Shell Diagrams

All internal lines <u>on-shell</u>





4 pt tree



4-pt 1-loop

NO "VIRTUAL PARTICLES"

Positive Grassmannian

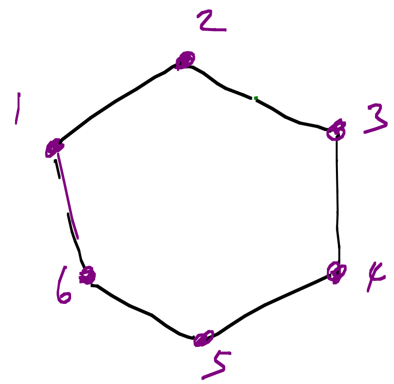
$G(k,n)$: k-planes in n-dimensions

$$C_{\alpha a} = \begin{array}{cc} & \xleftarrow{\quad n \quad} \\ \updownarrow k & \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \end{array} \Big/ GL(k)$$

$G^+(k,n)$: all minors $(a_1 \cdots a_k) > 0$

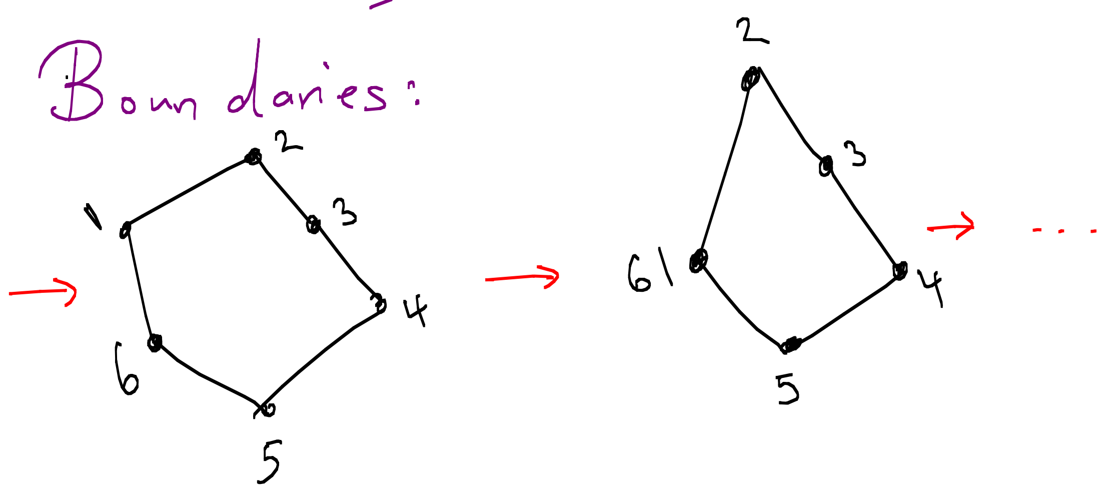
for $a_1 < a_2 \cdots < a_k$

$G^+(3,n)$



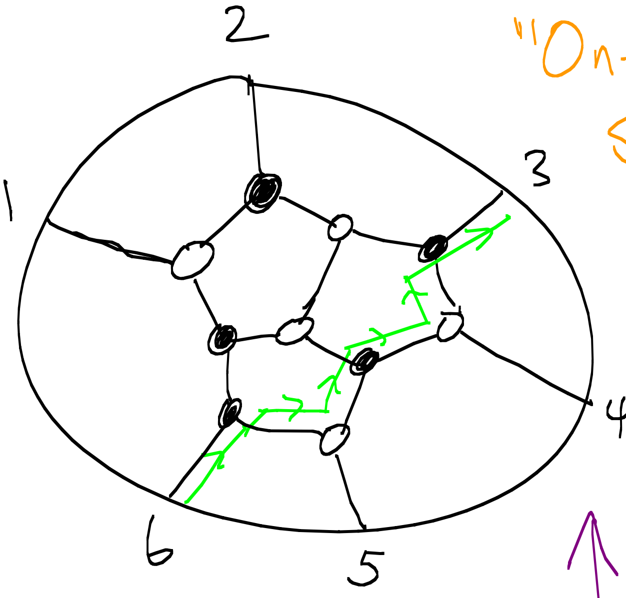
Convex Polygon

Boundaries:

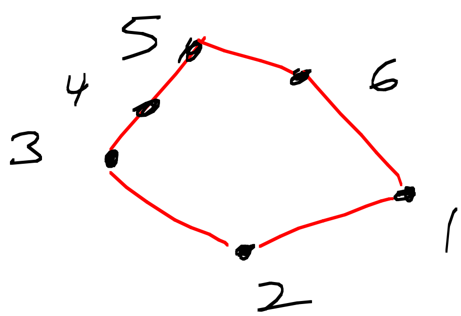


$1 \to 4$
$2 \to 6$
$3 \to 5$
$4 \to 7$
$5 \to 8$
$6 \to 9$

Affine Permutation



"On-shell" Spacetime Picture



Cell of Positive Grassmannian

Yangian Invariance

$\uparrow$

Positive Diffeomorphisms

* So: a beautiful story for building blocks of amplitude

– but why do we combine them as dictated by loop BCFW recursion?

To <u>make</u> result Local + Unitary!

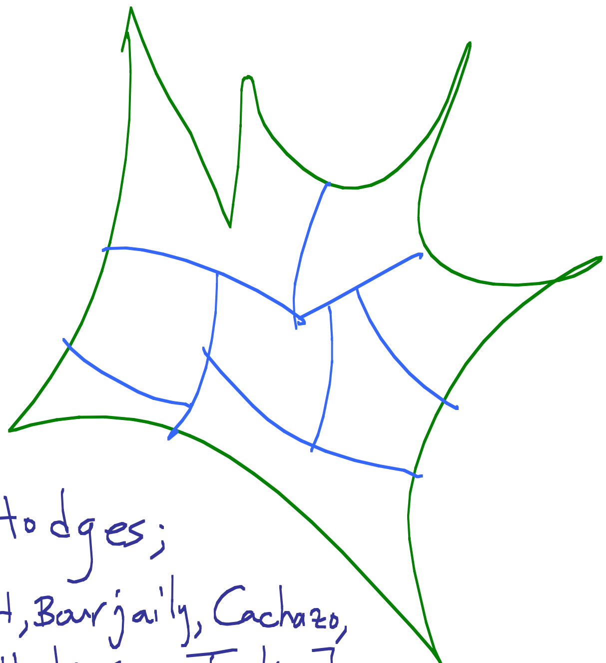
- Simplest example: NMHV tree

$$[1\,2\,3\,4\,5] = \int \frac{dc_1}{c_1} \cdots \frac{dc_5}{c_5} \Big/ GL(1) \; \delta^{4|4}(c_1 Z_1 + \cdots + c_5 Z_5) = \frac{\delta^4(\langle 1234\rangle \eta_5 + \cdots)}{\langle 1234\rangle \cdots \langle 5123\rangle}$$

- $$\mathcal{M}^{\text{tree}}_{BCFW} = \sum_{i<j} [1\;\, i\;\, i{+}1\;\, j\;\, j{+}1]$$

[WHY THIS COMBINATION?]

Vague Fantasy



[Hodges;
NAH, Bourjaily, Cachazo,
Hodges + Trnka]

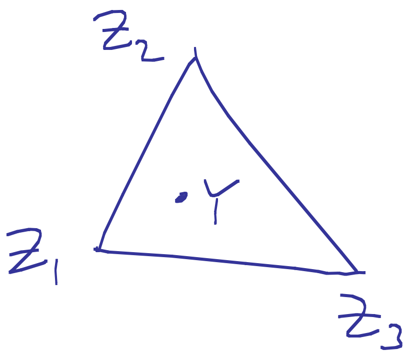
Amplitude is "the volume" of "some region" in "some space".

Many "triangulations".

Locality + Unitarity Emerge Somehow

Amplituhedron

Triangles $\longrightarrow$ Positive Grassmannian



$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I$$

$$(c_1 c_2 c_3)/GL(1), \quad c_a > 0$$

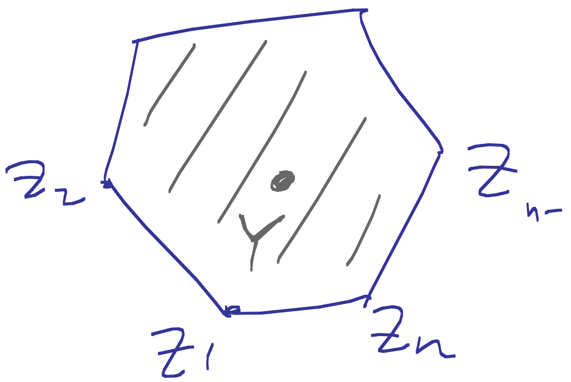
$$\longrightarrow \quad (c_1 \ldots c_n)/GL(1) \quad c_a > 0 \quad \text{Simplex}$$

$$\longrightarrow \quad \left(\vec{C}_1 \ldots \vec{C}_n\right)/GL(k), \quad (c_{a_1} \ldots c_{a_k}) > 0$$

$$a_1 < \ldots < a_k$$

Positive Grassmannian

Polygons



$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \rangle > 0$$

$$a_1 < a_2 < a_3$$

key point

$$Y^I = c_1 Z_1^I + \cdots + c_n Z_n^I$$

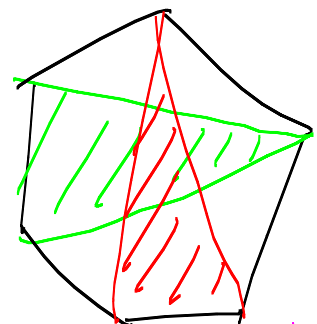
$(c_1 \dots c_n) \in G^+(1,n)$

$(Z_1, \dots, Z_n) \in G^+(3,n)$ $\longrightarrow$ Region in $G(1,3)$

$$Y^I = c_a Z_a^I$$

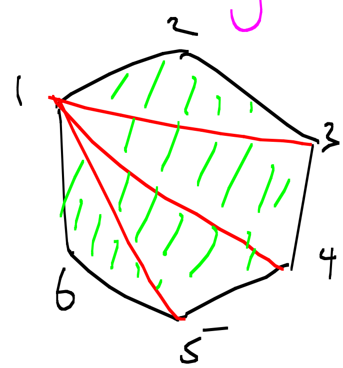
2D $\qquad\qquad$ $(n-1)$-d

* 2D cells of $G^+(1,n) \longrightarrow$

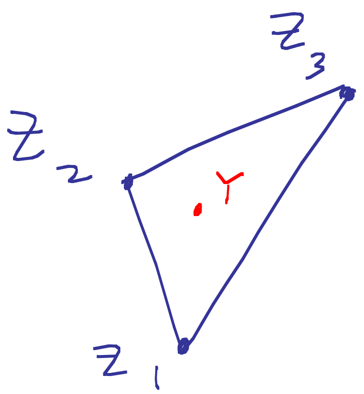


"Triangles"

* Triangulation, e.g $\sum_i (1\ i\ i{+}1)$

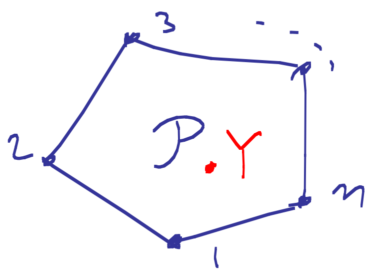


"Volume" as a Form



$$\Omega = \frac{\langle Y dY dY\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$= \frac{dx}{x}\frac{dy}{y}, \quad Y = Z_1 + xZ_2 + yZ_3$$



Ω_P : Form with log. sing. on ∂P. Immediate from triangulation.

[Form Ω_P makes sense for general complex Z's]

First Generalization

$$\boxed{Y^I = c_a Z_a^I}$$

$I = 1, \dots, \underset{\text{even}}{\underset{\uparrow}{l+m}}$

- $Z \in G^+(l+m, n)$, $c \in G^+(l, n)$

- Polygon : $m = 2$

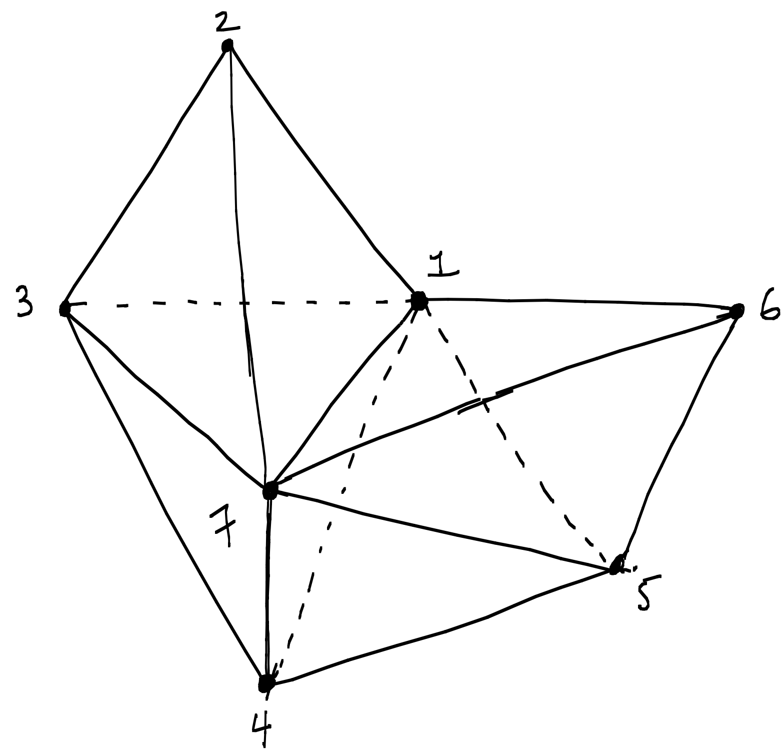
  First new case : $m = 4$

- Triangulation:

$$\sum_{i<j} (1\ i\ i{+}1\ j\ j{+}1)$$

- In fact: $M_{k=1}$ is "the volume" of this region!

A 3D "Face"



Tree Amplitude for $\left[1^- 2^+ 3^+ 4^+ 5^- 6^+ 7^- 8^-\right]$

$\{$Hundreds of Pages of Feynman Diagrams$\}$

Generalization

$$\boxed{Y_\alpha^I = C_{\alpha a} Z_a^I}$$

"Polygon" in $G(K,K+4)$

Positive Grassmannian $G^+(k,n)$

External Data in $G^+(4+K,n)$

New treatment of SUSY

$$\underbrace{\begin{pmatrix} Z_a \\ \hdashline \eta_a \end{pmatrix}}_{\mathbb{P}^{3|4}} \;\Bigg|\; \underbrace{\begin{array}{c} \\ 4 \\ k \end{array}\!\!\begin{array}{c} k \\ \begin{pmatrix} & 0 & & \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}}_{Y} , \underbrace{\begin{pmatrix} Z_a \\ \hdashline \phi_1 \cdot \eta_a \\ \vdots \\ \phi_k \cdot \eta_a \end{pmatrix}}_{\mathbb{P}^{3+k}} = \mathcal{Z}_a$$

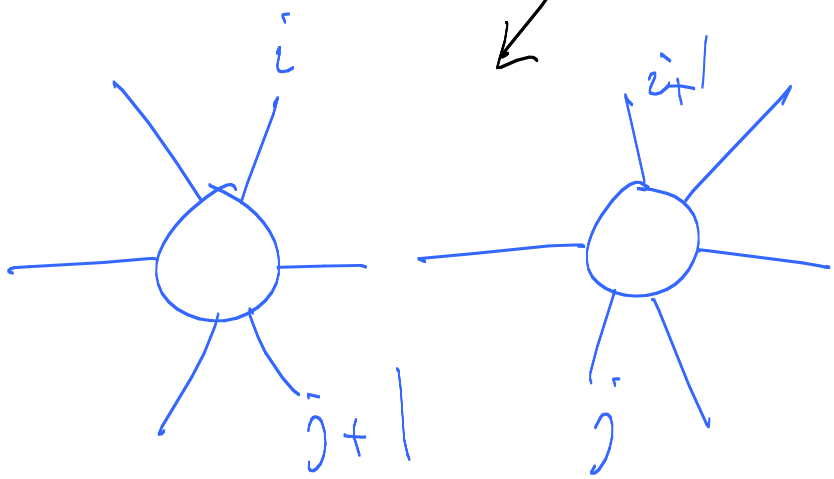
Super-Geometry $\longrightarrow$ Bosonic Geometry

$$\mathcal{M}_{n,k}[Z_a, \eta_a] = \int d^4\phi_1 \ldots d^4\phi_k \, M[\mathcal{Z}_a]$$

- Tree amplitude is "volume" of this region!
- BCFW is (one) "triangulation" [cell decomposition]

Boundaries of $P_{n,k}$

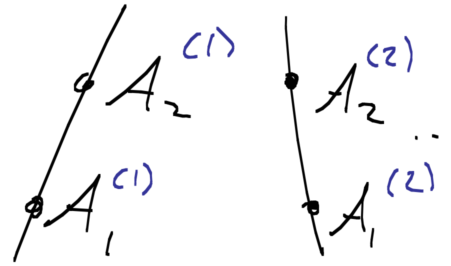
$$\begin{array}{c} \quad\quad\quad i \;\; i+1 \quad\quad\quad\quad\quad\quad j \;\; j+1 \quad\quad\quad \\ C = \begin{bmatrix} * & * & 0\;0\;0\;0 & * & * & 0\;0 \\ \multicolumn{3}{c}{\boxed{C_L}} & \multicolumn{3}{c}{0} \\ \multicolumn{3}{c}{0} & \multicolumn{3}{c}{\boxed{C_R}} \end{bmatrix} \end{array}$$



Unitarity From Positivity

Loops

$k = 0$: Lines in $\mathbb{P}^3$



$$A_\gamma^{I\,(i)} = Z^{(i)}_{\gamma a}\, Z_a^{I+}$$

$$\left[\begin{array}{c} Z^{(1)} \\ \hline Z^{(2)} \\ \hline \vdots \\ Z^{(L)} \end{array}\right]$$

All minors Positive!

Motivated by "Hiding Particles"

$$\begin{array}{c} \begin{array}{ccccc} A & B & C & D & 1 \;\; 2 \; \cdots \;\; n \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & Z^{(1)} \\ \cline{5-5} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & Z^{(2)} \end{array}\right] \end{array}$$

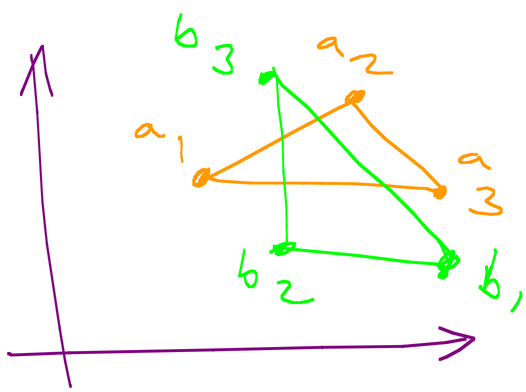
Simplest Case $n=4$

$$Z_{(i)} = \begin{bmatrix} 1 & x_i & 0 & -z_i \\ 0 & y_i & 1 & w_i \end{bmatrix}$$

$$\boxed{\vec{a}_i = \begin{pmatrix} x_i^+ \\ y_i^+ \end{pmatrix},\ \vec{b}_i = \begin{pmatrix} w_i^+ \\ z_i^+ \end{pmatrix},\ (\vec{a}_i - \vec{a}_j)\cdot(\vec{b}_i - \vec{b}_j) < 0}$$

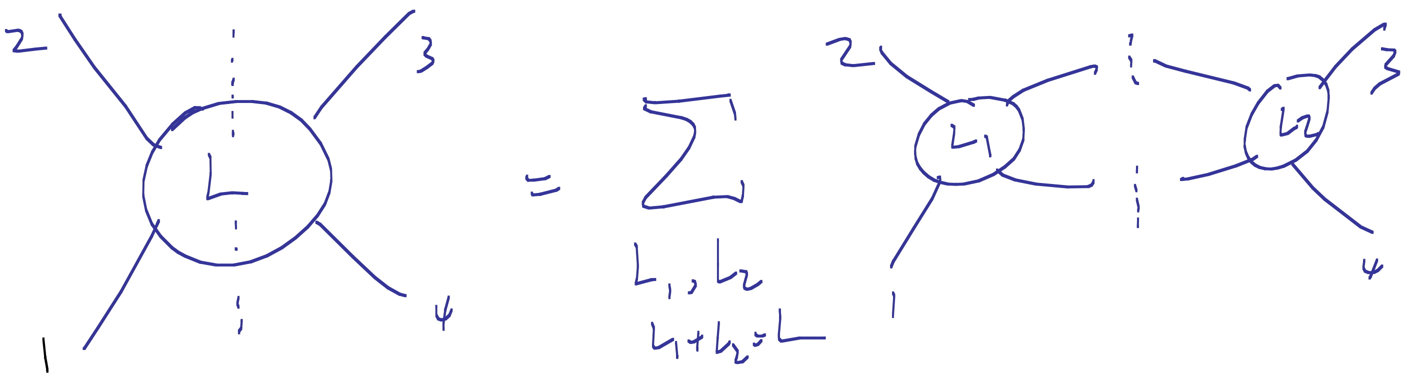
"Triangulate"

= 4-pt to all loop order!



A junior highschool geometry problem

Simple geometric identity from positivity



Textbook Unitarity from Positivity

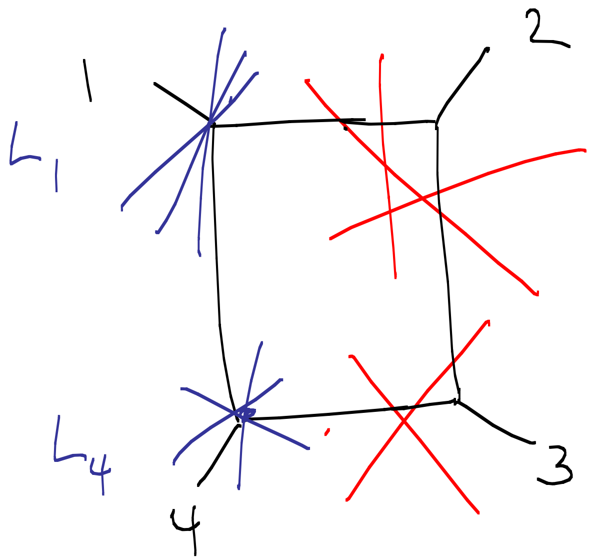
Towards the All-Loop Integrand

* Understand "faces" of Amplituhedron

  = compute cuts

* Many infinite classes have been explicitly triangulated/computed analytically

* First non-trivial, explicit, All-Loop order information on integrand!



1
2
3
4
L1
L4

$(u1^2\ u2\ v1^3\ v2^4\ w3\ w4\ w5+u1^3\ u2^3\ v1^3\ v2^4\ w3\ w4\ w5-u1\ u2\ v1^3\ v2^4\ w3^2\ w4$
$w5-u1^3\ u2^2\ v1^3\ v2^4\ w3^2\ w4\ w5-u1^2\ u2^3\ v1^3\ v2^4\ w3^2\ w4\ w5+u1^2\ u2^2\ v1^3\ v2^4\ w3^3$
$w4\ w5-u1\ u2\ v1^3\ v2^4\ w3\ w4^2\ w5-u1^3\ u2^2\ v1^3\ v2^4\ w3\ w4^2\ w5-u1^2\ u2^3\ v1^3\ v2^4\ w3$
$w4^2\ w5+u2\ v1^3\ v2^4\ w3^2\ w4^2\ w5+u1^3\ u2\ v1^3\ v2^4\ w3^2\ w4^2\ w5+2\ u1^2\ u2^2\ v1^3\ v2^4\ w3^2$
$w4^2\ w5+u1\ u2^3\ v1^3\ v2^4\ w3^2\ w4^2\ w5-u1^2\ u2\ v1^3\ v2^4\ w3^3\ w4^2\ w5-u1\ u2^2\ v1^3\ v2^4$
$w3^3\ w4^2\ w5+u1^2\ u2^2\ v1^3\ v2^4\ w3\ w4^3\ w5-u1^2\ u2\ v1^3\ v2^4\ w3^2\ w4^3\ w5-u1\ u2^2\ v1^3$
$v2^4\ w3^2\ w4^3\ w5+u1\ u2\ v1^3\ v2^4\ w3^3\ w4^3\ w5-u1^3\ u2^2\ v1^3\ v2^4\ w3\ w4\ w5^2+u1^3\ u2\ v1^3$
$v2^4\ w3^2\ w4\ w5^2+u1^2\ u2^2\ v1^3\ v2^4\ w3^2\ w4\ w5^2-u1^2\ u2\ v1^3\ v2^4\ w3^3\ w4\ w5^2+u1^3\ u2$
$v1^3\ v2^4\ w3\ w4^2\ w5^2+u1^2\ u2^2\ v1^3\ v2^4\ w3\ w4^2\ w5^2-u1^3\ v1^3\ v2^4\ w3^2\ w4^2\ w5^2-2\ u1^2$
$u2\ v1^3\ v2^4\ w3^2\ w4^2\ w5^2-u1\ u2^2\ v1^3\ v2^4\ w3^2\ w4^2\ w5^2+u1^2\ v1^3\ v2^4\ w3^3\ w4^2\ w5^2+u1$
$u2\ v1^3\ v2^4\ w3^3\ w4^2\ w5^2-u1^2\ u2\ v1^3\ v2^4\ w3\ w4^3\ w5^2+u1^2\ v1^3\ v2^4\ w3^2\ w4^3\ w5^2+u1$
$u2\ v1^3\ v2^4\ w3^2\ w4^3\ w5^2-u1\ v1^3\ v2^4\ w3^3\ w4^3\ w5^2-u1^3\ u2^3\ v1^3\ v2^3\ z3+u1^2\ u2^4\ v1^3$
$v2^3\ z3+u1^3\ u2^2\ v1^3\ v2^3\ w3\ z3-u1\ u2^4\ v1^3\ v2^3\ w3\ z3+u1^3\ u2^2\ v1^3\ v2^3\ w4\ z3-u1$
$u2^4\ v1^3\ v2^3\ w4\ z3-u1^3\ u2\ v1^3\ v2^3\ w3\ w4\ z3-u1^2\ u2^2\ v1^3\ v2^3\ w3\ w4\ z3+u1\ u2^3$
$v1^3\ v2^3\ w3\ w4\ z3+u2^4\ v1^3\ v2^3\ w3\ w4\ z3-u1^2\ u2^2\ v1^3\ v2^3\ w4^2\ z3+u1\ u2^3\ v1^3\ v2^3$
$w4^2\ z3+u1^2\ u2\ v1^3\ v2^3\ w3\ w4^2\ z3-u2^3\ v1^3\ v2^3\ w3\ w4^2\ z3+u1^3\ u2^2\ v1^3\ v2^3\ w5$
$z3-u1^2\ u2^3\ v1^3\ v2^3\ w5\ z3-u1^3\ u2\ v1^3\ v2^3\ w3\ w5\ z3+u1\ u2^3\ v1^3\ v2^3\ w3\ w5\ z3-$
$u1^3\ u2\ v1^3\ v2^3\ w4\ w5\ z3+u1\ u2^3\ v1^3\ v2^3\ w4\ w5\ z3+u1^2\ u2\ v1^3\ v2^3\ w3\ w4\ w5$
$z3-u1\ u2^2\ v1^3\ v2^3\ w3\ w4\ w5\ z3-u2^3\ v1^3\ v2^3\ w3\ w4\ w5\ z3-u1^3\ u2^3\ v1^3\ v2^3\ w3$
$w4\ w5\ z3+u1^3\ v1^2\ v2^4\ w3\ w4\ w5\ z3-u1^2\ u2\ v1^2\ v2^4\ w3\ w4\ w5\ z3-u1^3\ u2^3\ v1^2$
$v2^4\ w3\ w4\ w5\ z3+u1^2\ u2^3\ v1^3\ v2^3\ w3^2\ w4\ w5\ z3+u1^3\ u2^2\ v1^2\ v2^4\ w3^2\ w4\ w5\ z3$
$+u1^2\ u2\ v1^3\ v2^3\ w4^2\ w5\ z3-u1\ u2^2\ v1^3\ v2^3\ w4^2\ w5\ z3+u2^2\ v1^3\ v2^3\ w3\ w4^2\ w5\ z3$
$+u1^3\ u2^2\ v1^3\ v2^3\ w3\ w4^2\ w5\ z3+u1^2\ u2^3\ v1^3\ v2^3\ w3\ w4^2\ w5\ z3-u1^2\ v1^2\ v2^4\ w3$
$w4^2\ w5\ z3+u1\ u2\ v1^2\ v2^4\ w3\ w4^2\ w5\ z3+u1^3\ u2^2\ v1^2\ v2^4\ w3\ w4^2\ w5\ z3+u1^2\ u2^3$
$v1^2\ v2^4\ w3\ w4^2\ w5\ z3-u1^2\ u2^2\ v1^3\ v2^3\ w3^2\ w4^2\ w5\ z3-u1\ u2^3\ v1^3\ v2^3\ w3^2\ w4^2\ w5$
$z3-u1^3\ u2\ v1^2\ v2^4\ w3^2\ w4^2\ w5\ z3-u1^2\ u2^2\ v1^2\ v2^4\ w3^2\ w4^2\ w5\ z3-u1^2\ u2^2\ v1^3$
$v2^3\ w3\ w4^3\ w5\ z3-u1^2\ u2^2\ v1^2\ v2^4\ w3\ w4^3\ w5\ z3+u1\ u2^2\ v1^3\ v2^3\ w3^2\ w4^3\ w5\ z3$
$+u1^2\ u2\ v1^2\ v2^4\ w3^2\ w4^3\ w5\ z3+u1^3\ u2^2\ v1^3\ v2^3\ w3\ w4\ w5^2\ z3+u1^3\ u2^2\ v1^2\ v2^4$
$w3\ w4\ w5^2\ z3-u1^2\ u2^2\ v1^3\ v2^3\ w3^2\ w4\ w5^2\ z3-u1^3\ u2\ v1^2\ v2^4\ w3^2\ w4\ w5^2\ z3-u1^3$
$u2\ v1^3\ v2^3\ w3\ w4^2\ w5^2\ z3-u1^2\ u2^2\ v1^3\ v2^3\ w3\ w4^2\ w5^2\ z3-u1^3\ u2\ v1^2\ v2^4\ w3\ w4^2$
$w5^2\ z3-u1^2\ u2^2\ v1^2\ v2^4\ w3\ w4^2\ w5^2\ z3+u1^2\ u2\ v1^3\ v2^3\ w3^2\ w4^2\ w5^2\ z3+u1\ u2^2$
$v1^3\ v2^3\ w3^2\ w4^2\ w5^2\ z3+u1^3\ v1^2\ v2^4\ w3^2\ w4^2\ w5^2\ z3+u1^2\ u2\ v1^2\ v2^4\ w3^2\ w4^2\ w5^2$
$z3+u1^2\ u2\ v1^3\ v2^3\ w3\ w4^3\ w5^2\ z3+u1^2\ u2\ v1^2\ v2^4\ w3\ w4^3\ w5^2\ z3-u1\ u2\ v1^3\ v2^3$
$w3^2\ w4^3\ w5^2\ z3-u1^2\ v1^2\ v2^4\ w3^2\ w4^3\ w5^2\ z3+u1^3\ u2^3\ v1^2\ v2^3\ z3^2-u1^2\ u2^4\ v1^2\ v2^3$
$z3^2-u1^3\ u2^2\ v1^2\ v2^3\ w4\ z3^2+u1\ u2^4\ v1^2\ v2^3\ w4\ z3^2+u1^2\ u2^2\ v1^2\ v2^3\ w4^2\ z3^2-u1$
$u2^3\ v1^2\ v2^3\ w4^2\ z3^2-u1^3\ u2^2\ v1^2\ v2^3\ w5\ z3^2+u1^2\ u2^3\ v1^2\ v2^3\ w5\ z3^2+u1^3\ u2\ v1^2$
$v2^3\ w4\ w5\ z3^2-u1\ u2^3\ v1^2\ v2^3\ w4\ w5\ z3^2+u1^3\ u2^3\ v1^2\ v2^3\ w3\ w4\ w5\ z3^2-u1^2\ u2$
$v1^2\ v2^3\ w4^2\ w5\ z3^2+u1\ u2^2\ v1^2\ v2^3\ w4^2\ w5\ z3^2-$

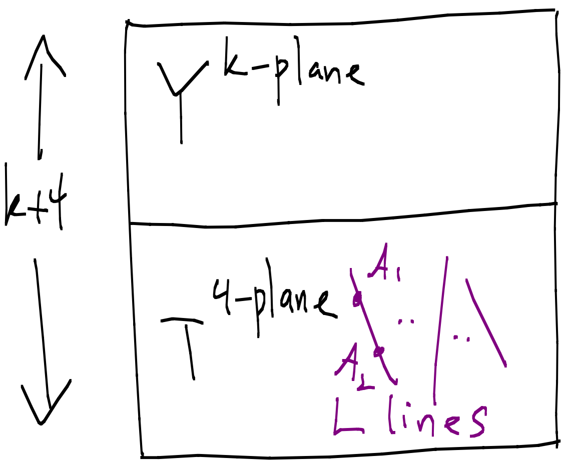
$u1\ u2^2\ v1^3\ v2^3\ w3\ w4\ w5\ z4-u1^3\ u2^3\ v1^3\ v2^3\ w3\ w4\ w5\ z4-u1^3\ u2^3\ v1^2\ v2^4$
$w3\ w4\ w5\ z4+u2^2\ v1^3\ v2^3\ w3^2\ w4\ w5\ z4+u1^3\ u2^2\ v1^3\ v2^3\ w3^2\ w4\ w5\ z4+u1^2\ u2^3$
$v1^3\ v2^3\ w3^2\ w4\ w5\ z4-u1^2\ v1^2\ v2^4\ w3^2\ w4\ w5\ z4+u1\ u2\ v1^2\ v2^4\ w3^2\ w4\ w5\ z4$
$+u1^3\ u2^2\ v1^2\ v2^4\ w3^2\ w4\ w5\ z4+u1^2\ u2^3\ v1^2\ v2^4\ w3^2\ w4\ w5\ z4-u1^2\ u2^2\ v1^3\ v2^3$
$w3^3\ w4\ w5\ z4-u1^2\ u2^2\ v1^2\ v2^4\ w3^3\ w4\ w5\ z4+u1^2\ u2^3\ v1^3\ v2^3\ w3\ w4^2\ w5\ z4+u1^3$
$u2^2\ v1^2\ v2^4\ w3\ w4^2\ w5\ z4-u1^2\ u2^2\ v1^3\ v2^3\ w3^2\ w4^2\ w5\ z4-u1\ u2^3\ v1^3\ v2^3\ w3^2$
$w4^2\ w5\ z4-u1^3\ u2\ v1^2\ v2^4\ w3^2\ w4^2\ w5\ z4-u1^2\ u2^2\ v1^2\ v2^4\ w3^2\ w4^2\ w5\ z4+u1$
$u2^2\ v1^3\ v2^3\ w3^3\ w4^2\ w5\ z4+u1^2\ u2\ v1^2\ v2^4\ w3^3\ w4^2\ w5\ z4+u1^3\ u2^2\ v1^3\ v2^3\ w3$
$w4\ w5^2\ z4+u1^3\ u2^2\ v1^2\ v2^4\ w3\ w4\ w5^2\ z4-u1^3\ u2\ v1^3\ v2^3\ w3^2\ w4\ w5^2\ z4-u1^2$
$u2^2\ v1^3\ v2^3\ w3^2\ w4\ w5^2\ z4-u1^3\ u2\ v1^2\ v2^4\ w3^2\ w4\ w5^2\ z4-u1^2\ u2^2\ v1^2\ v2^4\ w3^2$
$w4\ w5^2\ z4+u1^2\ u2\ v1^3\ v2^3\ w3^3\ w4\ w5^2\ z4+u1^2\ u2\ v1^2\ v2^4\ w3^3\ w4\ w5^2\ z4-u1^2$
$u2^2\ v1^3\ v2^3\ w3\ w4^2\ w5^2\ z4-u1^3\ u2\ v1^2\ v2^4\ w3\ w4^2\ w5^2\ z4+u1^2\ u2\ v1^3\ v2^3\ w3^2$
$w4^2\ w5^2\ z4+u1\ u2^2\ v1^3\ v2^3\ w3^2\ w4^2\ w5^2\ z4+u1^3\ v1^2\ v2^4\ w3^2\ w4^2\ w5^2\ z4+u1^2\ u2$
$v1^2\ v2^4\ w3^2\ w4^2\ w5^2\ z4-u1\ u2\ v1^3\ v2^3\ w3^3\ w4^2\ w5^2\ z4-u1^2\ v1^2\ v2^4\ w3^3\ w4^2\ w5^2$
$z4+u1^3\ u2^3\ v1^3\ v2^2\ z3\ z4-u1^2\ u2^4\ v1^3\ v2^2\ z3\ z4+u1^3\ u2^3\ v1^2\ v2^3\ z3\ z4-u1^2$
$u2^4\ v1^2\ v2^3\ z3\ z4-u1^2\ u2^3\ v1^3\ v2^2\ w3\ z3\ z4+u1\ u2^4\ v1^3\ v2^2\ w3\ z3\ z4-2\ u1^3$
$u2^2\ v1^2\ v2^3\ w3\ z3\ z4+u1^2\ u2^3\ v1^2\ v2^3\ w3\ z3\ z4+u1\ u2^4\ v1^2\ v2^3\ w3\ z3\ z4-u1^2$
$u2^3\ v1^3\ v2^2\ w4\ z3\ z4+u1\ u2^4\ v1^3\ v2^2\ w4\ z3\ z4-u1^3\ u2^2\ v1^2\ v2^3\ w4\ z3\ z4+u1^2$
$u2^3\ v1^2\ v2^3\ w4\ z3\ z4+u1^2\ u2^2\ v1^3\ v2^2\ w3\ w4\ z3\ z4-u2^4\ v1^3\ v2^2\ w3\ w4\ z3\ z4$
$+u1^3\ u2\ v1^2\ v2^3\ w3\ w4\ z3\ z4+u1^2\ u2^2\ v1^2\ v2^3\ w3\ w4\ z3\ z4-2\ u1\ u2^3\ v1^2\ v2^3$
$w3\ w4\ z3\ z4-u1^3\ u2^2\ v1^3\ v2^2\ w5\ z3\ z4+u1^2\ u2^3\ v1^3\ v2^2\ w5\ z3\ z4-u1^3\ u2^2\ v1^2$
$v2^3\ w5\ z3\ z4+u1^2\ u2^3\ v1^2\ v2^3\ w5\ z3\ z4+u1^2\ u2^2\ v1^3\ v2^2\ w3\ w5\ z3\ z4-u1\ u2^3$
$v1^3\ v2^2\ w3\ w5\ z3\ z4+2\ u1^3\ u2\ v1^2\ v2^3\ w3\ w5\ z3\ z4-u1^2\ u2^2\ v1^2\ v2^3\ w3\ w5\ z3$
$z4-u1\ u2^3\ v1^2\ v2^3\ w3\ w5\ z3\ z4+u1^2\ u2^2\ v1^3\ v2^2\ w4\ w5\ z3\ z4-u1\ u2^3\ v1^3\ v2^2$
$w4\ w5\ z3\ z4+u1^3\ u2\ v1^2\ v2^3\ w4\ w5\ z3\ z4-u1^2\ u2^2\ v1^2\ v2^3\ w4\ w5\ z3\ z4+u2^3$
$v1^3\ v2^2\ w3\ w4\ w5\ z3\ z4+u1^3\ u2^3\ v1^3\ v2^2\ w3\ w4\ w5\ z3\ z4-2\ u1^2\ u2\ v1^2\ v2^3\ w3$
$w4\ w5\ z3\ z4+2\ u1\ u2^2\ v1^2\ v2^3\ w3\ w4\ w5\ z3\ z4+2\ u1^3\ u2^3\ v1^2\ v2^3\ w3\ w4\ w5$
$z3\ z4-u1^3\ v1\ v2^4\ w3\ w4\ w5\ z3\ z4+u1^2\ u2\ v1\ v2^4\ w3\ w4\ w5\ z3\ z4+u1^3\ u2^3\ v1$
$v2^4\ w3\ w4\ w5\ z3\ z4-u1^2\ u2^3\ v1^3\ v2^2\ w3^2\ w4\ w5\ z3\ z4-u1^3\ u2^2\ v1^2\ v2^3\ w3^2\ w4$
$w5\ z3\ z4-u1^2\ u2^3\ v1^2\ v2^3\ w3^2\ w4\ w5\ z3\ z4-u1^3\ u2^2\ v1\ v2^4\ w3^2\ w4\ w5\ z3\ z4-$
$u1^2\ u2^3\ v1^3\ v2^2\ w3\ w4^2\ w5\ z3\ z4-u1^3\ u2^2\ v1^2\ v2^3\ w3\ w4^2\ w5\ z3\ z4-u1^2\ u2^3$
$v1^2\ v2^3\ w3\ w4^2\ w5\ z3\ z4-u1^3\ u2^2\ v1\ v2^4\ w3\ w4^2\ w5\ z3\ z4+u1\ u2^3\ v1^3\ v2^2\ w3^2$
$w4^2\ w5\ z3\ z4+2\ u1^2\ u2^2\ v1^2\ v2^3\ w3^2\ w4^2\ w5\ z3\ z4+u1^3\ u2\ v1\ v2^4\ w3^2\ w4^2\ w5$
$z3\ z4-u1^3\ u2^2\ v1^3\ v2^2\ w3\ w4\ w5^2\ z3\ z4-2\ u1^3\ u2^2\ v1^2\ v2^3\ w3\ w4\ w5^2\ z3\ z4-$
$u1^3\ u2^2\ v1\ v2^4\ w3\ w4\ w5^2\ z3\ z4+u1^2\ u2^2\ v1^3\ v2^2\ w3^2\ w4\ w5^2\ z3\ z4+u1^3\ u2\ v1^2$
$v2^3\ w3^2\ w4\ w5^2\ z3\ z4+u1^2\ u2^2\ v1^2\ v2^3\ w3^2\ w4\ w5^2\ z3\ z4+u1^3\ u2\ v1\ v2^4\ w3^2\ w4$
$w5^2\ z3\ z4+u1^2\ u2^2\ v1^3\ v2^2\ w3\ w4^2\ w5^2\ z3\ z4+u1^3\ u2\ v1^2\ v2^3\ w3\ w4^2\ w5^2\ z3\ z4$
$+u1^2\ u2^2\ v1^2\ v2^3\ w3\ w4^2\ w5^2\ z3\ z4+u1^3\ u2\ v1\ v2^4\ w3\ w4^2\ w5^2\ z3\ z4-u1\ u2^2$
$v1^3\ v2^2\ w3^2\ w4^2\ w5^2\ z3\ z4-2\ u1^2\ u2\ v1^2\ v2^3\ w3^2\ w4^2\ w5^2\ z3\ z4-u1^3\ v1\ v2^4$
$w3^2\ w4^2\ w5^2\ z3\ z4-u1^3\ u2^3\ v1^2\ v2^2\ z3^2\ z4+u1^2\ u2^4\ v1^2\ v2^2\ z3^2\ z4-u1^3\ u2^3\ v1$
$v2^3\ z3^2\ z4+u1^2\ u2^4\ v1\ v2^3\ z3^2\ z4+$

$u1^2 u2^3 v1^2 v2^2 w4 z3^2 z4-u1 u2^4 v1^2 v2^2 w4 z3^2 z4+u1^3 u2^2 v1$
$v2^3 w4 z3^2 z4-u1^2 u2^3 v1 v2^3 w4 z3^2 z4+u1^3 u2^2 v1^2 v2^2 w5 z3^2$
$z4-u1^2 u2^3 v1^2 v2^2 w5 z3^2 z4+u1^3 u2^2 v1 v2^3 w5 z3^2 z4-u1^2 u2^3 v1$
$v2^3 w5 z3^2 z4-u1^2 u2^2 v1^2 v2^2 w4 w5 z3^2 z4+u1 u2^3 v1^2 v2^2 w4 w5$
$z3^2 z4-u1^3 u2 v1 v2^3 w4 w5 z3^2 z4+u1^2 u2^2 v1 v2^3 w4 w5 z3^2 z4-$
$u1^3 u2^3 v1^2 v2^2 w3 w4 w5 z3^2 z4-u1^3 u2^3 v1 v2^3 w3 w4 w5 z3^2 z4$
$+u1^2 u2^3 v1^2 v2^2 w3 w4^2 w5 z3^2 z4+u1^3 u2^2 v1 v2^3 w3 w4^2 w5 z3^2$
$z4+u1^3 u2^2 v1^2 v2^2 w3 w4 w5^2 z3^2 z4+u1^3 u2^2 v1 v2^3 w3 w4 w5^2 z3^2$
$z4-u1^2 u2^2 v1^2 v2^2 w3 w4^2 w5^2 z3^2 z4-u1^3 u2 v1 v2^3 w3 w4^2 w5^2 z3^2$
$z4+u1^2 u2^2 v1^2 v2^3 w3^2 z4^2-u1 u2^3 v1^2 v2^3 w3^2 z4^2-u1^2 u2 v1^2 v2^3$
$w3^2 w5 z4^2+u1 u2^2 v1^2 v2^3 w3^2 w5 z4^2+u1^3 u2^3 v1^2 v2^3 w3 w4 w5$
$z4^2-u1^3 u2^2 v1^2 v2^3 w3^2 w4 w5 z4^2-u1^2 u2^3 v1^2 v2^3 w3^2 w4 w5 z4^2$
$+u1^2 u2^2 v1^2 v2^3 w3^3 w4 w5 z4^2-u1^3 u2^2 v1^2 v2^3 w3 w4 w5^2 z4^2+u1^3$
$u2 v1^2 v2^3 w3^2 w4 w5^2 z4^2+u1^2 u2^2 v1^2 v2^3 w3^2 w4 w5^2 z4^2-u1^2 u2$
$v1^2 v2^3 w3^3 w4 w5^2 z4^2-u1^3 u2^3 v1^2 v2^2 z3 z4^2+u1^2 u2^4 v1^2 v2^2 z3$
$z4^2+u1^2 u2^3 v1^2 v2^2 w3 z3 z4^2-u1 u2^4 v1^2 v2^2 w3 z3 z4^2+u1^3 u2^2 v1$
$v2^3 w3 z3 z4^2-u1^2 u2^3 v1 v2^3 w3 z3 z4^2+u1^3 u2^2 v1^2 v2^2 w5 z3$
$z4^2-u1^2 u2^3 v1^2 v2^2 w5 z3 z4^2-u1^2 u2^2 v1^2 v2^2 w3 w5 z3 z4^2+u1 u2^3$
$v1^2 v2^2 w3 w5 z3 z4^2-u1^3 u2 v1 v2^3 w3 w5 z3 z4^2+u1^2 u2^2 v1 v2^3$
$w3 w5 z3 z4^2-u1^3 u2^3 v1^2 v2^2 w3 w4 w5 z3 z4^2-u1^3 u2^3 v1 v2^3 w3$
$w4 w5 z3 z4^2+u1^2 u2^3 v1^2 v2^2 w3^2 w4 w5 z3 z4^2+u1^3 u2^2 v1 v2^3 w3^2$
$w4 w5 z3 z4^2+u1^3 u2^2 v1^2 v2^2 w3 w4 w5^2 z3 z4^2+u1^3 u2^2 v1 v2^3 w3$
$w4 w5^2 z3 z4^2-u1^2 u2^2 v1^2 v2^2 w3^2 w4 w5^2 z3 z4^2-u1^3 u2 v1 v2^3 w3^2$
$w4 w5^2 z3 z4^2+u1^3 u2^3 v1 v2^2 z3^2 z4^2-u1^2 u2^4 v1 v2^2 z3^2 z4^2-u1^3$
$u2^2 v1 v2^2 w5 z3^2 z4^2+u1^2 u2^3 v1 v2^2 w5 z3^2 z4^2+u1^3 u2^3 v1 v2^2 w3$
$w4 w5 z3^2 z4^2-u1^3 u2^2 v1 v2^2 w3 w4 w5^2 z3^2 z4^2+u1^2 u2^2 v1^3 v2^3 w3$
$w4 z5-u1 u2^3 v1^3 v2^3 w3 w4 z5-u1 u2 v1^3 v2^3 w3^2 w4^2 z5+u2^2 v1^3$
$v2^3 w3^2 w4^2 z5+u1 u2^2 v1^3 v2^3 w3 w4 w5 z5-u1^3 u2^3 v1^3 v2^3 w3 w4$
$w5 z5-u1^2 u2 v1^2 v2^4 w3 w4 w5 z5+u1^3 u2^2 v1^3 v2^3 w3^2 w4 w5 z5$
$+u1^2 u2^3 v1^3 v2^3 w3^2 w4 w5 z5-u1^2 u2^2 v1^3 v2^3 w3^3 w4 w5 z5+u1^3$
$u2^2 v1^3 v2^3 w3 w4^2 w5 z5+u1^2 u2^3 v1^3 v2^3 w3 w4^2 w5 z5-u2 v1^3 v2^3$
$w3^2 w4^2 w5 z5-u1^3 u2 v1^3 v2^3 w3^2 w4^2 w5 z5-2 u1^2 u2^2 v1^3 v2^3 w3^2$
$w4^2 w5 z5-u1 u2^3 v1^3 v2^3 w3^2 w4^2 w5 z5+u1 v1^2 v2^4 w3^2 w4^2 w5 z5$
$+u1^2 u2 v1^3 v2^3 w3^3 w4^2 w5 z5+u1 u2^2 v1^3 v2^3 w3^3 w4^2 w5 z5-u1^2$
$u2^2 v1^3 v2^3 w3 w4^3 w5 z5+u1^2 u2 v1^3 v2^3 w3^2 w4^3 w5 z5+u1 u2^2 v1^3$
$v2^3 w3^2 w4^3 w5 z5-u1 u2 v1^3 v2^3 w3^3 w4^3 w5 z5+u1^3 u2^3 v1^3 v2^2 z3$
$z5-2 u1^2 u2^4 v1^3 v2^2 z3 z5+u1^3 u2^3 v1^2 v2^3 z3 z5-u1^3 u2^2 v1^3 v2^2$
$w3 z3 z5+u1^2 u2^3 v1^3 v2^2 w3 z3 z5+2 u1 u2^4 v1^3 v2^2 w3 z3 z5-u1^3$
$u2^2 v1^2 v2^3 w3 z3 z5-$

$u_1^2 u_2^3 v_1^2 v_2^2 w_4 z_3^2 z_4-u_1 u_2^4 v_1^2 v_2^2 w_4 z_3^2 z_4+u_1^3 u_2^2 v_1$
$v_2^3 w_4 z_3^2 z_4-u_1^2 u_2^3 v_1 v_2^3 w_4 z_3^2 z_4+u_1^3 u_2^2 v_1^2 v_2^2 w_5 z_3^2$
$z_4-u_1^2 u_2^3 v_1^2 v_2^2 w_5 z_3^2 z_4+u_1^3 u_2^2 v_1 v_2^3 w_5 z_3^2 z_4-u_1^2 u_2^3 v_1$
$v_2^3 w_5 z_3^2 z_4-u_1^2 u_2^2 v_1^2 v_2^2 w_4 w_5 z_3^2 z_4+u_1 u_2^3 v_1^2 v_2^2 w_4 w_5$
$z_3^2 z_4-u_1^3 u_2 v_1 v_2^3 w_4 w_5 z_3^2 z_4+u_1^2 u_2^2 v_1 v_2^3 w_4 w_5 z_3^2 z_4-$
$u_1^3 u_2^3 v_1^2 v_2^2 w_3 w_4 w_5 z_3^2 z_4-u_1^3 u_2^3 v_1 v_2^3 w_3 w_4 w_5 z_3^2 z_4$
$+u_1^2 u_2^3 v_1^2 v_2^2 w_3 w_4^2 w_5 z_3^2 z_4+u_1^3 u_2^2 v_1 v_2^3 w_3 w_4^2 w_5 z_3^2$
$z_4+u_1^3 u_2^2 v_1^2 v_2^2 w_3 w_4 w_5^2 z_3^2 z_4+u_1^3 u_2^2 v_1 v_2^3 w_3 w_4 w_5^2 z_3^2$
$z_4-u_1^2 u_2^2 v_1^2 v_2^2 w_3 w_4^2 w_5^2 z_3^2 z_4-u_1^3 u_2 v_1 v_2^3 w_3 w_4^2 w_5^2 z_3^2$
$z_4+u_1^2 u_2^2 v_1^2 v_2^3 w_3^2 z_4^2-u_1 u_2^3 v_1^2 v_2^3 w_3^2 z_4^2-u_1^2 u_2 v_1^2 v_2^3$
$w_3^2 w_5 z_4^2+u_1 u_2^2 v_1^2 v_2^3 w_3^2 w_5 z_4^2+u_1^3 u_2^3 v_1^2 v_2^3 w_3 w_4 w_5$
$z_4^2-u_1^3 u_2^2 v_1^2 v_2^3 w_3^2 w_4 w_5 z_4^2-u_1^2 u_2^3 v_1^2 v_2^3 w_3^2 w_4 w_5 z_4^2$
$+u_1^2 u_2^2 v_1^2 v_2^3 w_3^3 w_4 w_5 z_4^2-u_1^3 u_2^2 v_1^2 v_2^3 w_3 w_4 w_5^2 z_4^2+u_1^3$
$u_2 v_1^2 v_2^3 w_3^2 w_4 w_5^2 z_4^2+u_1^2 u_2^2 v_1^2 v_2^3 w_3^2 w_4 w_5^2 z_4^2-u_1^2 u_2$
$v_1^2 v_2^3 w_3^3 w_4 w_5^2 z_4^2-u_1^3 u_2^3 v_1^2 v_2^2 z_3 z_4^2+u_1^2 u_2^4 v_1^2 v_2^2 z_3$
$z_4^2+u_1^2 u_2^3 v_1^2 v_2^2 w_3 z_3 z_4^2-u_1 u_2^4 v_1^2 v_2^2 w_3 z_3 z_4^2+u_1^3 u_2^2 v_1$
$v_2^3 w_3 z_3 z_4^2-u_1^2 u_2^3 v_1 v_2^3 w_3 z_3 z_4^2+u_1^3 u_2^2 v_1^2 v_2^2 w_5 z_3$
$z_4^2-u_1^2 u_2^3 v_1^2 v_2^2 w_5 z_3 z_4^2-u_1^2 u_2^2 v_1^2 v_2^2 w_3 w_5 z_3 z_4^2+u_1 u_2^3$
$v_1^2 v_2^2 w_3 w_5 z_3 z_4^2-u_1^3 u_2 v_1 v_2^3 w_3 w_5 z_3 z_4^2+u_1^2 u_2^2 v_1 v_2^3$
$w_3 w_5 z_3 z_4^2-u_1^3 u_2^3 v_1^2 v_2^2 w_3 w_4 w_5 z_3 z_4^2-u_1^3 u_2^3 v_1 v_2^3 w_3$
$w_4 w_5 z_3 z_4^2+u_1^2 u_2^3 v_1^2 v_2^2 w_3^2 w_4 w_5 z_3 z_4^2+u_1^3 u_2^2 v_1 v_2^3 w_3^2$
$w_4 w_5 z_3 z_4^2+u_1^3 u_2^2 v_1^2 v_2^2 w_3 w_4 w_5^2 z_3 z_4^2+u_1^3 u_2^2 v_1 v_2^3 w_3$
$w_4 w_5^2 z_3 z_4^2-u_1^2 u_2^2 v_1^2 v_2^2 w_3^2 w_4 w_5^2 z_3 z_4^2-u_1^3 u_2 v_1 v_2^3 w_3^2$
$w_4 w_5^2 z_3 z_4^2+u_1^3 u_2^3 v_1 v_2^2 z_3^2 z_4^2-u_1^2 u_2^4 v_1 v_2^2 z_3^2 z_4^2-u_1^3$
$u_2^2 v_1 v_2^2 w_5 z_3^2 z_4^2+u_1^2 u_2^3 v_1 v_2^2 w_5 z_3^2 z_4^2+u_1^3 u_2^3 v_1 v_2^2 w_3$
$w_4 w_5 z_3^2 z_4^2-u_1^3 u_2^2 v_1 v_2^2 w_3 w_4 w_5^2 z_3^2 z_4^2+u_1^2 u_2^2 v_1^3 v_2^3 w_3$
$w_4 z_5-u_1 u_2^3 v_1^3 v_2^3 w_3 w_4 z_5-u_1 u_2 v_1^3 v_2^3 w_3^2 w_4^2 z_5+u_2^2 v_1^3$
$v_2^3 w_3^2 w_4^2 z_5+u_1 u_2^2 v_1^3 v_2^3 w_3 w_4 w_5 z_5-u_1^3 u_2^3 v_1^3 v_2^3 w_3 w_4$
$w_5 z_5-u_1^2 u_2 v_1^2 v_2^4 w_3 w_4 w_5 z_5+u_1^3 u_2^2 v_1^3 v_2^3 w_3^2 w_4 w_5 z_5$
$+u_1^2 u_2^3 v_1^3 v_2^3 w_3^2 w_4 w_5 z_5-u_1^2 u_2^2 v_1^3 v_2^3 w_3^3 w_4 w_5 z_5+u_1^3$
$u_2^2 v_1^3 v_2^3 w_3 w_4^2 w_5 z_5+u_1^2 u_2^3 v_1^3 v_2^3 w_3 w_4^2 w_5 z_5-u_2 v_1^3 v_2^3$
$w_3^2 w_4^2 w_5 z_5-u_1^3 u_2 v_1^3 v_2^3 w_3^2 w_4^2 w_5 z_5-2 u_1^2 u_2^2 v_1^3 v_2^3 w_3^2$
$w_4^2 w_5 z_5-u_1 u_2^3 v_1^3 v_2^3 w_3^2 w_4^2 w_5 z_5+u_1 v_1^2 v_2^4 w_3^2 w_4^2 w_5 z_5$
$+u_1^2 u_2 v_1^3 v_2^3 w_3^3 w_4^2 w_5 z_5+u_1 u_2^2 v_1^3 v_2^3 w_3^3 w_4^2 w_5 z_5-u_1^2$
$u_2^2 v_1^3 v_2^3 w_3 w_4^3 w_5 z_5+u_1^2 u_2 v_1^3 v_2^3 w_3^2 w_4^3 w_5 z_5+u_1 u_2^2 v_1^3$
$v_2^3 w_3^2 w_4^3 w_5 z_5-u_1 u_2 v_1^3 v_2^3 w_3^3 w_4^3 w_5 z_5+u_1^3 u_2^3 v_1^3 v_2^2 z_3$
$z_5-2 u_1^2 u_2^4 v_1^3 v_2^2 z_3 z_5+u_1^3 u_2^3 v_1^2 v_2^3 z_3 z_5-u_1^3 u_2^2 v_1^3 v_2^2$
$w_3 z_3 z_5+u_1^2 u_2^3 v_1^3 v_2^2 w_3 z_3 z_5+2 u_1 u_2^4 v_1^3 v_2^2 w_3 z_3 z_5-u_1^3$
$u_2^2 v_1^2 v_2^3 w_3 z_3 z_5-$

$u1^{2} u2 v1 v2^{3} w4^{2} w5 z3^{2} z5+u1^{3} u2^{2} v1^{2} v2^{2} w3 w4^{2} w5 z3^{2} z5+u1^{2} u2^{3} v1^{2}$
$v2^{2} w3 w4^{2} w5 z3^{2} z5-u1^{2} u2^{2} v1^{2} v2^{2} w3 w4^{3} w5 z3^{2} z5+u1^{2} u2^{2} v1^{3} v2^{2} w3^{2}$
$z4 z5-2 u1 u2^{3} v1^{3} v2^{2} w3^{2} z4 z5+u1^{2} u2^{2} v1^{2} v2^{3} w3^{2} z4 z5-u1 u2^{3} v1^{3} v2^{2}$
$w3 w4 z4 z5+u1^{2} u2^{2} v1^{2} v2^{3} w3 w4 z4 z5+2 u2^{3} v1^{3} v2^{2} w3^{2} w4 z4 z5-2 u1^{2}$
$u2 v1^{2} v2^{3} w3^{2} w4 z4 z5+u1 u2^{2} v1^{3} v2^{2} w3^{2} w5 z4 z5-u1^{2} u2 v1^{2} v2^{3} w3^{2} w5$
$z4 z5+u1^{3} u2^{3} v1^{3} v2^{2} w3 w4 w5 z4 z5+u1^{3} u2^{3} v1^{2} v2^{3} w3 w4 w5 z4 z5-u2^{2}$
$v1^{3} v2^{2} w3^{2} w4 w5 z4 z5-u1^{3} u2^{2} v1^{3} v2^{2} w3^{2} w4 w5 z4 z5-u1^{2} u2^{3} v1^{3} v2^{2}$
$w3^{2} w4 w5 z4 z5-u1^{3} u2^{2} v1^{2} v2^{3} w3^{2} w4 w5 z4 z5-u1^{2} u2^{3} v1^{2} v2^{3} w3^{2} w4 w5$
$z4 z5+u1^{2} v1 v2^{4} w3^{2} w4 w5 z4 z5+u1^{2} u2^{2} v1^{3} v2^{2} w3^{3} w4 w5 z4 z5+u1^{2} u2^{2}$
$v1^{2} v2^{3} w3^{3} w4 w5 z4 z5-u1^{2} u2^{3} v1^{3} v2^{2} w3 w4^{2} w5 z4 z5-u1^{3} u2^{2} v1^{2} v2^{3}$
$w3 w4^{2} w5 z4 z5+u1^{2} u2^{2} v1^{3} v2^{2} w3^{2} w4^{2} w5 z4 z5+u1 u2^{3} v1^{3} v2^{2} w3^{2} w4^{2}$
$w5 z4 z5+u1^{3} u2 v1^{2} v2^{3} w3^{2} w4^{2} w5 z4 z5+u1^{2} u2^{2} v1^{2} v2^{3} w3^{2} w4^{2} w5 z4$
$z5-u1 u2^{2} v1^{3} v2^{2} w3^{3} w4^{2} w5 z4 z5-u1^{2} u2 v1^{2} v2^{3} w3^{3} w4^{2} w5 z4 z5-u1^{3}$
$u2^{3} v1^{3} v2 z3 z4 z5+2 u1^{2} u2^{4} v1^{3} v2 z3 z4 z5-2 u1^{3} u2^{3} v1^{2} v2^{2} z3 z4 z5$
$+2 u1^{2} u2^{4} v1^{2} v2^{2} z3 z4 z5-u1^{3} u2^{3} v1 v2^{3} z3 z4 z5+u1^{2} u2^{3} v1^{3} v2 w3 z3$
$z4 z5-2 u1 u2^{4} v1^{3} v2 w3 z3 z4 z5+2 u1^{3} u2^{2} v1^{2} v2^{2} w3 z3 z4 z5-2 u1^{2}$
$u2^{3} v1^{2} v2^{2} w3 z3 z4 z5-2 u1 u2^{4} v1^{2} v2^{2} w3 z3 z4 z5+2 u1^{3} u2^{2} v1 v2^{3} w3$
$z3 z4 z5+u1^{2} u2^{3} v1 v2^{3} w3 z3 z4 z5+u1^{2} u2^{3} v1^{3} v2 w4 z3 z4 z5-2 u1 u2^{4}$
$v1^{3} v2 w4 z3 z4 z5+u1^{3} u2^{2} v1^{2} v2^{2} w4 z3 z4 z5-u1^{2} u2^{3} v1^{2} v2^{2} w4 z3 z4$
$z5+u1^{3} u2^{2} v1 v2^{3} w4 z3 z4 z5+2 u2^{4} v1^{3} v2 w3 w4 z3 z4 z5-2 u1^{2} u2^{2} v1^{2}$
$v2^{2} w3 w4 z3 z4 z5+3 u1 u2^{3} v1^{2} v2^{2} w3 w4 z3 z4 z5-2 u1^{3} u2 v1 v2^{3} w3$
$w4 z3 z4 z5-u1^{2} u2^{2} v1 v2^{3} w3 w4 z3 z4 z5-u1^{2} u2^{3} v1^{3} v2 w5 z3 z4 z5+u1^{3}$
$u2^{2} v1^{2} v2^{2} w5 z3 z4 z5-u1^{2} u2^{3} v1^{2} v2^{2} w5 z3 z4 z5+u1^{3} u2^{2} v1 v2^{3} w5 z3$
$z4 z5+u1 u2^{3} v1^{3} v2 w3 w5 z3 z4 z5+u1^{2} u2^{2} v1^{2} v2^{2} w3 w5 z3 z4 z5+u1 u2^{3}$
$v1^{2} v2^{2} w3 w5 z3 z4 z5-2 u1^{3} u2 v1 v2^{3} w3 w5 z3 z4 z5-u1^{2} u2^{2} v1 v2^{3} w3$
$w5 z3 z4 z5+u1 u2^{3} v1^{3} v2 w4 w5 z3 z4 z5-u1^{3} u2 v1 v2^{3} w4 w5 z3 z4 z5-$
$u2^{3} v1^{3} v2 w3 w4 w5 z3 z4 z5-u1^{3} u2^{3} v1^{3} v2 w3 w4 w5 z3 z4 z5-u1 u2^{2} v1^{2}$
$v2^{2} w3 w4 w5 z3 z4 z5-2 u1^{3} u2^{3} v1^{2} v2^{2} w3 w4 w5 z3 z4 z5+u1^{2} u2 v1 v2^{3}$
$w3 w4 w5 z3 z4 z5-u1^{3} u2^{3} v1 v2^{3} w3 w4 w5 z3 z4 z5+u1^{3} v2^{4} w3 w4 w5 z3$
$z4 z5+u1^{2} u2^{3} v1^{3} v2 w3^{2} w4 w5 z3 z4 z5+u1^{3} u2^{2} v1^{2} v2^{2} w3^{2} w4 w5 z3 z4$
$z5+u1^{2} u2^{3} v1^{2} v2^{2} w3^{2} w4 w5 z3 z4 z5+u1^{3} u2^{2} v1 v2^{3} w3^{2} w4 w5 z3 z4 z5$
$+u1^{2} u2^{3} v1^{3} v2 w3 w4^{2} w5 z3 z4 z5+u1^{3} u2^{2} v1^{2} v2^{2} w3 w4^{2} w5 z3 z4 z5+u1^{2}$
$u2^{3} v1^{2} v2^{2} w3 w4^{2} w5 z3 z4 z5+u1^{3} u2^{2} v1 v2^{3} w3 w4^{2} w5 z3 z4 z5-u1 u2^{3}$
$v1^{3} v2 w3^{2} w4^{2} w5 z3 z4 z5-2 u1^{2} u2^{2} v1^{2} v2^{2} w3^{2} w4^{2} w5 z3 z4 z5-u1^{3} u2$
$v1 v2^{3} w3^{2} w4^{2} w5 z3 z4 z5+u1^{3} u2^{3} v1^{2} v2 z3^{2} z4 z5-2 u1^{2} u2^{4} v1^{2} v2 z3^{2}$
$z4 z5+2 u1^{3} u2^{3} v1 v2^{2} z3^{2} z4 z5-2 u1^{2} u2^{4} v1 v2^{2} z3^{2} z4 z5+u1^{3} u2^{3} v2^{3}$
$z3^{2} z4 z5-u1^{2} u2^{3} v1^{2} v2 w4 z3^{2} z4 z5+2 u1 u2^{4} v1^{2} v2 w4 z3^{2} z4 z5-u1^{3}$
$u2^{2} v1 v2^{2} w4 z3^{2} z4 z5+u1^{2} u2^{3} v1 v2^{2} w4 z3^{2} z4 z5-u1^{3} u2^{2} v2^{3} w4 z3^{2} z4$
$z5+u1^{2} u2^{3} v1^{2} v2 w5 z3^{2} z4 z5-u1^{3} u2^{2} v1 v2^{2} w5 z3^{2} z4 z5+u1^{2} u2^{3} v1 v2^{2}$
$w5 z3^{2} z4 z5-u1^{3} u2^{2} v2^{3} w5 z3^{2} z4 z5-u1 u2^{3} v1^{2} v2 w4 w5 z3^{2} z4 z5+u1^{3}$
$u2 v2^{3} w4 w5 z3^{2} z4 z5+u1^{3} u2^{3} v1^{2} v2 w3 w4 w5 z3^{2} z4 z5+u1^{3} u2^{3} v1 v2^{2}$
$w3 w4 w5 z3^{2} z4 z5-u1^{2} u2^{3} v1^{2} v2 w3 w4^{2} w5 z3^{2} z4 z5-u1^{3} u2^{2} v1 v2^{2} w3$
$w4^{2} w5 z3^{2} z4 z5-u1^{2} u2^{2} v1^{2} v2^{2} w3^{2} z4^{2} z5+2 u1 u2^{3} v1^{2} v2^{2} w3^{2} z4^{2} z5-$
$u1^{2} u2^{2} v1 v2^{3} w3^{2} z4^{2} z5-u1 u2^{2} v1^{2} v2^{2} w3^{2} w5 z4^{2} z5+u1^{2} u2 v1 v2^{3} w3^{2}$
$w5 z4^{2} z5-u1^{3} u2^{3} v1^{2} v2^{2} w3 w4 w5 z4^{2} z5+u1^{3} u2^{2} v1^{2} v2^{2} w3^{2} w4 w5 z4^{2}$
$z5+u1^{2} u2^{3} v1^{2} v2^{2} w3^{2} w4 w5 z4^{2} z5-u1^{2} u2^{2} v1^{2} v2^{2} w3^{3} w4 w5 z4^{2} z5+u1^{3}$
$u2^{3} v1^{2} v2 z3 z4^{2} z5-2 u1^{2} u2^{4} v1^{2} v2 z3 z4^{2} z5+u1^{3} u2^{3} v1 v2^{2} z3 z4^{2} z5-$
$u1^{2} u2^{3} v1^{2} v2 w3 z3 z4^{2} z5+2 u1 u2^{4} v1^{2} v2 w3 z3 z4^{2} z5-u1^{3} u2^{2} v1 v2^{2}$
$w3 z3 z4^{2} z5+u1^{2} u2^{3} v1 v2^{2} w3 z3 z4^{2} z5-u1^{3} u2^{2} v2^{3} w3 z3 z4^{2} z5+u1^{2} u2^{3}$
$v1^{2} v2 w5 z3 z4^{2} z5-u1^{3} u2^{2} v1 v2^{2} w5 z3 z4^{2} z5-u1 u2^{3} v1^{2} v2 w3 w5 z3$
$z4^{2} z5+u1^{3} u2 v2^{3} w3 w5 z3 z4^{2} z5+u1^{3} u2^{3} v1^{2} v2 w3 w4 w5 z3 z4^{2} z5+u1^{3}$
$u2^{3} v1 v2^{2} w3 w4 w5 z3 z4^{2} z5-u1^{2} u2^{3} v1^{2} v2 w3^{2} w4 w5 z3 z4^{2} z5-u1^{3} u2^{2}$
$v1 v2^{2} w3^{2} w4 w5 z3 z4^{2} z5-u1^{3} u2^{3} v1 v2 z3^{2} z4^{2} z5+2 u1^{2} u2^{4} v1 v2 z3^{2}$

Amplituhedron $\mathcal{P}_{n,k,L}$

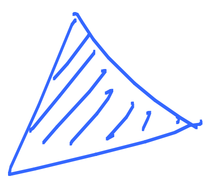
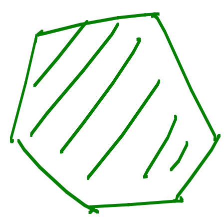


$$Y_\alpha^{I} = C_{\alpha a} Z_a^{+I}$$

$$\mathcal{A}_\gamma^{(i)} = \mathcal{D}^{(i)}_{\gamma a} Z_a^{+I}$$

$$\begin{array}{c} \leftarrow n \rightarrow \\ \left(\begin{array}{c} C \\ \hline \mathcal{D}^{(1)} \\ \hline \vdots \\ \mathcal{D}^{(L)} \end{array} \right) \end{array}$$

All minors w/ C > 0

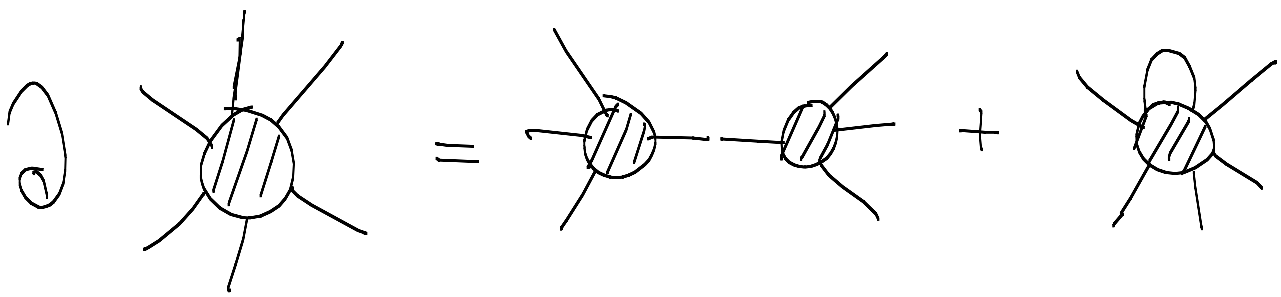
Generalizes  ,  , "hidden particles"

- $\Omega_{n,k,L}$: top-form w/ log sing. on $\partial \mathcal{P}_{n,k,L}$

$\rightarrow$ Obtain by triangulating $\mathcal{P}_{n,k,L}$

- $$\mathcal{M}_{n,k,L} = \int d^4\phi_1 \ldots d^4\phi_k \, \Omega_{n,k,L} \, \delta^{4k}(Y; Y_0).$$

$$Y_0 = \left(\begin{array}{ccc|cccc} 1 & & & 0 & 0 & 0 & 0 \\ & \ddots & & & & \vdots & \\ & & 1 & 0 & 0 & 0 & 0 \end{array}\right), \quad Z_a = \begin{pmatrix} Z_a \\ \hline \phi_1 \cdot \eta_a \\ \vdots \\ \phi_k \cdot \eta_a \end{pmatrix}$$



Follows From Positive Geometry

- All symmetries manifest
- Determining integrand reduced to "triangulating" $\mathcal{P}_{n,k,L}$

[BCFW one triangulation – not "best" one beyond tree level]

* This simple mathematical structure gives a complete, autonomous definition of all scattering amplitudes in Planar $\mathcal{N}=4$ SYM, totally free of usual QFT language: no Feynman Diagrams, not even on-shell diagrams, recursion relations etc.

(Momentum) Twistor Space : Kinematics

Grassmannian Positivity : Dynamics

We Now Have

A first example of emergent locality + unitarity, [space-time + quantum mechanics]

* This structure is as of yet un-known to the mathematicians — surprising that it can exist [+ well beyond positive Grassmannian]

General Comments/ Speculations

Beyond Dualities

* Not equivalence between existing gauge-redundant descriptions

* We are finding <u>new</u> reprentations guaranteeing  without manifest "local evolution through spacetime".

Gravity, Strings?

* Unexplained magic even in planar SYM
  (identities between different color orderings,
  Bern-Carrasco-Johansson relations, ..)
* Similar combinatorial properties in BCFW

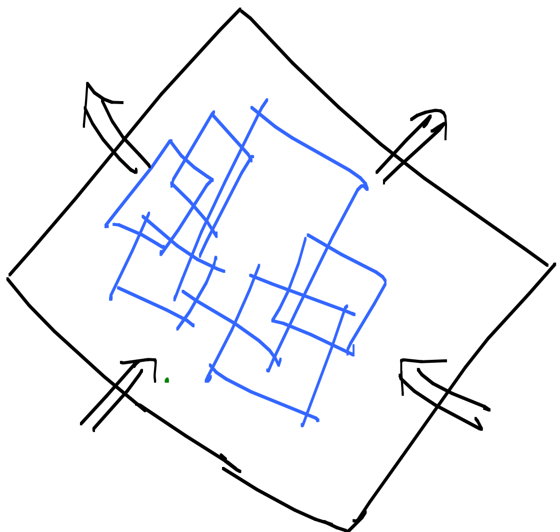
Not just "Emergent Spacetime"

Space-Time + Quantum Mechanics

should emerge together, inextricably

linked [as in our baby $\mathcal{N}=4$ example]

Wild Fantasy



* Bulk theory of "quantum" "gravity"

emergent

emergent

"Triangulations" ~ "Covering with Diamonds"