2d $\mathcal{N} = (2,2)$ supersymmetry with $U(1)_{R_V}$ in curved space

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Summary

Based on:

- C. Closset, SC, *to appear*.
- F. Benini, SC, "Partition functions of $\mathcal{N} = (2,2)$ gauge theories on S^2 and vortices", arXiv:1206.2356 [hep-th].

What we do:

- Define $\mathcal{N} = (2,2)$ field theories on compact Riemann surfaces
- Classify supersymmetric backgrounds

Why:

- Exact results by localization
- Type II strings on Calabi-Yau
- 4d $\mathcal{N} = 2$ field theories

Plan

1. 2d $\mathcal{N} = (2,2)$ SUSY in flat space
2. 2d $\mathcal{N} = (2,2)$ SUSY with $U(1)_{R_V}$ in curved space
   - Killing spinor equations
   - Supersymmetry algebra
   - Supermultiplets
   - Supersymmetric Lagrangians
3. Supersymmetric backgrounds
4. Outlook

2d $\mathcal{N} = (2, 2)$ SUSY in flat space

2d $\mathcal{N} = (2,2)$ on flat $\mathbb{R}^2$

[Hori: Trieste Lectures on Mirror Symmetry, 2002]

Superalgebra

$$\{Q_\pm, \widetilde{Q}_\pm\} = -P_{\pm\pm} \qquad \{Q_\mp, \widetilde{Q}_\pm\} = 0$$

$$\{Q_-, Q_+\} = \widetilde{Z} \qquad \{\widetilde{Q}_-, \widetilde{Q}_+\} = Z$$

$$\{Q_-, \widetilde{Q}_+\} = W \qquad \{\widetilde{Q}_-, Q_+\} = \widetilde{W}$$

$$[M, Q_\mp] = \pm\tfrac{1}{2} Q_\mp \qquad [M, \widetilde{Q}_\mp] = \pm\tfrac{1}{2}\widetilde{Q}_\mp$$

$U(1)_{R_A} \times U(1)_{R_V}$ automorphism

$$[R_V, Q_\mp] = -Q_\mp \qquad [R_V, \widetilde{Q}_\mp] = +\widetilde{Q}_\mp$$

$$[R_A, Q_\mp] = \pm Q_\mp \qquad [R_A, \widetilde{Q}_\mp] = \mp\widetilde{Q}_\mp$$

$$[R_V, Z] = 2Z \qquad [R_V, \widetilde{Z}] = -2\widetilde{Z}$$

$$[R_A, W] = 2W \qquad [R_A, \widetilde{W}] = -2\widetilde{W}$$

$$[R_V, W] = [R_V, \widetilde{W}] = 0 \qquad [R_A, Z] = [R_A, \widetilde{Z}] = 0$$

$\mathbb{Z}_2$ mirror automorphism: $\theta^- \leftrightarrow -\widetilde{\theta}^-$

$$Q_- \leftrightarrow \widetilde{Q}_- \,, \qquad R_V \leftrightarrow R_A \,, \qquad Z \leftrightarrow W \,, \qquad \widetilde{Z} \leftrightarrow \widetilde{W}$$

2d $\mathcal{N} = (2,2)$ on flat $\mathbb{R}^2$: supermultiplets

- **Chiral Multiplet** Φ: $\widetilde{D}_-\Phi = \widetilde{D}_+\Phi = 0$.

  Can have nontrivial R_V, W, $\widetilde{W}$ charges.

- **Vector Multiplet**: $\mathcal{V} = \widetilde{\mathcal{V}} \sim \mathcal{V} + \Phi + \widetilde{\Phi}$.

- **Twisted Chiral Multiplet** Ω: $D_-\Omega = \widetilde{D}_+\Omega = 0$. [Gates, Hull, Roček 1984]

  Can have nontrivial R_A, Z, $\widetilde{Z}$ charges.

- **Twisted Vector Multiplet**: $\widehat{\mathcal{V}} = \widetilde{\widehat{\mathcal{V}}} \sim \widehat{\mathcal{V}} + \Omega + \widetilde{\Omega}$.

2d $\mathcal{N} = (2,2)$ with $U(1)_{R_V}$ on flat $\mathbb{R}^2$

$Z = \widetilde{Z} = 0$, Axial R-symmetry not required.

Flat space superalgebra

$$
\begin{aligned}
&\{Q_\pm, \widetilde{Q}_\pm\} = -P_{\pm\pm} && \{Q_\mp, \widetilde{Q}_\pm\} = 0 \\
&\{Q_-, Q_+\} = 0 && \{\widetilde{Q}_-, \widetilde{Q}_+\} = 0 \\
&\{Q_-, \widetilde{Q}_+\} = W && \{\widetilde{Q}_-, Q_+\} = \widetilde{W} \\
&[M, Q_\mp] = \pm\tfrac{1}{2} Q_\mp && [M, \widetilde{Q}_\mp] = \pm\tfrac{1}{2}\widetilde{Q}_\mp \\
&[R_V, Q_\mp] = -Q_\mp && [R_V, \widetilde{Q}_\mp] = +\widetilde{Q}_\mp \\
&[R_V, W] = 0 && [R_V, \widetilde{W}] = 0
\end{aligned}
$$

We will generalize this supersymmetry algebra to curved space *à la* [Festuccia, Seiberg 2011].

2d $\mathcal{N} = (2,2)$ SUSY with $U(1)_{R_V}$ in curved space

Coupling to background supergravity

Couple the 2d $\mathcal{R}$-multiplet [Dumitrescu, Seiberg 2011] to external new minimal SUGRA multiplet, similarly to [Closset, Dumitrescu, Festuccia, Komargodski 2012] in 3d.

Supercurrent $\mathcal{R}$-multiplet	$T^{\mu\nu}$	$S^{\mu\alpha}$	$j^{\mu}_{R_V}$	G_μ	$\widetilde{G}_\mu$
New min SUGRA multiplet	$h_{\mu\nu}$	$\psi_{\mu\alpha}$	A_μ	$\widetilde{C}_\mu$	C_μ

- $j^{\mu}_{R_V}$, $-\varepsilon^{\mu\nu}\widetilde{G}_\nu$, $\varepsilon^{\mu\nu}G_\nu$: conserved currents for R_V, W, $\widetilde{W}$.
- A_μ, C_μ, $\widetilde{C}_\mu$: conjugate gauge fields.

Killing spinor equations

- Frame: $\{e^1 = \sqrt{g}^{\frac{1}{2}}\, dz,\; e^{\bar{1}} = \sqrt{g}^{\frac{1}{2}}\, d\bar{z}\}$
- Metric: $ds^2 = \sqrt{g}\, dz\, d\bar{z}$

Killing spinor equations

$$(\nabla_1 - i A_1)\zeta_- = 0 \qquad (\nabla_{\bar{1}} - i A_{\bar{1}})\zeta_- = \tfrac{1}{2}\mathcal{H}\,\zeta_+$$
$$(\nabla_1 - i A_1)\zeta_+ = \tfrac{1}{2}\widetilde{\mathcal{H}}\,\zeta_- \qquad (\nabla_{\bar{1}} - i A_{\bar{1}})\zeta_+ = 0$$

$$(\nabla_1 + i A_1)\widetilde{\zeta}_- = 0 \qquad (\nabla_{\bar{1}} + i A_{\bar{1}})\widetilde{\zeta}_- = \tfrac{1}{2}\widetilde{\mathcal{H}}\,\widetilde{\zeta}_+$$
$$(\nabla_1 + i A_1)\widetilde{\zeta}_+ = \tfrac{1}{2}\mathcal{H}\,\widetilde{\zeta}_- \qquad (\nabla_{\bar{1}} + i A_{\bar{1}})\widetilde{\zeta}_+ = 0$$

where $\widetilde{\mathcal{H}} = -i\varepsilon^{\mu\nu}\partial_\mu \widetilde{C}_\nu$, $\mathcal{H} = -i\varepsilon^{\mu\nu}\partial_\mu C_\nu$.

2d $\mathcal{N} = (2,2)$ with $U(1)_{R_V}$ in curved space

Curved space supersymmetry algebra

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\varphi_{(r,w,\widetilde{w})} = -2i\left(\mathcal{L}'_K + \zeta_+\widetilde{\zeta}_-\left(w - \frac{r}{2}\mathcal{H}\right) - \zeta_-\widetilde{\zeta}_+\left(\widetilde{w} - \frac{r}{2}\widetilde{\mathcal{H}}\right)\right)\varphi_{(r,w,\widetilde{w})}$$

$$\{\delta_\zeta, \delta_\eta\}\varphi_{(r,w,\widetilde{w})} = 0 \qquad \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\eta}}\}\varphi_{(r,w,\widetilde{w})} = 0$$

where

$$\mathcal{L}'_K\varphi_{(r,w,\widetilde{w})} = \left(K^\mu D_\mu + \tfrac{i}{2}(D_\mu K_\nu)S^{\mu\nu}\right)\varphi_{(r,w,\widetilde{w})}$$

is the totally covariant Lie derivative along the Killing vector $K^\mu = \zeta\gamma^\mu\widetilde{\zeta}$,

$$D_\mu\varphi_{(r,w,\widetilde{w})} \equiv \left(\nabla_\mu - irA_\mu + \tfrac{1}{2}(w\widetilde{C}_\mu - \widetilde{w}C_\mu)\right)\varphi_{(r,w,\widetilde{w})}$$

the totally covariant derivative of a field $\varphi_{(r,w,\widetilde{w})}$.

Supersymmetry transformations

Can be written for a general multiplet. *E.g.*:

Chiral multiplet of charges $(r, w, \widetilde{w})$

$$
\begin{aligned}
\delta\phi &= \sqrt{2}(\zeta_+\psi_- - \zeta_-\psi_+) \\
\delta\psi_- &= \sqrt{2}\zeta_- F - \sqrt{2}i\widetilde{\zeta}_- \left(w - \frac{r}{2}\mathcal{H}\right)\phi + 2\sqrt{2}i\widetilde{\zeta}_+ D_1\phi \\
\delta\psi_+ &= \sqrt{2}\zeta_+ F - \sqrt{2}i\widetilde{\zeta}_+ \left(\widetilde{w} - \frac{r}{2}\widetilde{\mathcal{H}}\right)\phi + 2\sqrt{2}i\widetilde{\zeta}_- D_{\bar{1}}\phi \\
\delta F &= \sqrt{2}i\left(\widetilde{w} - \frac{r-2}{2}\widetilde{\mathcal{H}}\right)\widetilde{\zeta}_+\psi_- - \sqrt{2}i\left(w - \frac{r-2}{2}\mathcal{H}\right)\widetilde{\zeta}_-\psi_+ \\
&\quad + 2\sqrt{2}iD_1(\widetilde{\zeta}_+\psi_+) - 2\sqrt{2}iD_{\bar{1}}(\widetilde{\zeta}_-\psi_-)
\end{aligned}
$$

Supersymmetric Lagrangians

Kinetic Lagrangian

$$\mathcal{L}_{\widetilde{\Phi}\Phi} = 4D_1\widetilde{\phi}D_{\bar{1}}\phi - \widetilde{F}F + 2i\widetilde{\psi}_+D_1\psi_+ - 2i\widetilde{\psi}_-D_{\bar{1}}\psi_- +$$
$$+\left(w - \frac{r}{2}\mathcal{H}\right)\left(\widetilde{w} - \frac{r}{2}\widetilde{\mathcal{H}}\right)\widetilde{\phi}\phi - \frac{r}{4}R\widetilde{\phi}\phi + \frac{1}{2}\left(\widetilde{w}\mathcal{H} + w\widetilde{\mathcal{H}}\right)\widetilde{\phi}\phi +$$
$$+ i\left(\widetilde{w} - \frac{r}{2}\widetilde{\mathcal{H}}\right)\widetilde{\psi}_+\psi_- - i\left(w - \frac{r}{2}\mathcal{H}\right)\widetilde{\psi}_-\psi_+$$

F-term Lagrangian from superpotential $W(\Phi^i)$ of $r = 2$, $w = \widetilde{w} = 0$

$$\mathcal{L}_{F+\widetilde{F}} = F^i\partial_iW + \psi_-^i\psi_+^j\partial_i\partial_jW + \widetilde{F}^{\bar{i}}\partial_{\bar{i}}\widetilde{W} + \widetilde{\psi}_+^{\bar{i}}\widetilde{\psi}_-^{\bar{j}}\partial_{\bar{i}}\partial_{\bar{j}}\widetilde{W}$$

Similarly vector, twisted chiral ([Gomis, Lee 2012] on S^2), twisted vector multiplet.

Supersymmetric backgrounds

Supersymmetric backgrounds: 1 supercharge ζ

R-symmetry line bundle L: $m \equiv c_1(L) = \frac{1}{2\pi}\int_{\Sigma_g} dA = n(g-1) \in \mathbb{Z}$.

R_V-charge 1 Killing spinors

$$\zeta_-(z,\bar z) = \lambda s_0 \sqrt{g}^{\frac{1+n}{4}} f_1(\bar z)\,, \qquad \zeta_+(z,\bar z) = \frac{\lambda}{s_0}\sqrt{g}^{\frac{1-n}{4}} f_2(z)$$

$$\overline{f_1(\bar z)} \in \Gamma(\sqrt{K}^{-1-n})\,, \qquad f_2(z) \in \Gamma(\sqrt{K}^{-1+n})$$

$\overline{f_1(\bar z)}$, $f_2(z)$: *regular* holomorphic sections. λ, s_0: non-vanishing functions.

$U(1)_{R_V}$ connection A

$$A_z = \tfrac{n}{2}\omega_z - i\partial_z \ln(\lambda s_0)\,, \qquad A_{\bar z} = \tfrac{n}{2}\omega_{\bar z} - i\partial_{\bar z}\ln(\lambda/s_0)$$

$\mathcal{H}$, $\widetilde{\mathcal{H}}$

$$\partial_{\bar z} f_1(\bar z) + \tfrac{1+n}{2} f_1(\bar z)\,\partial_{\bar z}\ln\sqrt{g} + f_1(\bar z)\partial_{\bar z}\ln s_0^2 = \tfrac{1}{2}\mathcal{H} f_2(z)\,\tfrac{1}{s_0^2}\sqrt{g}^{\frac{1-n}{2}}$$

$$\partial_z f_2(z) + \tfrac{1-n}{2} f_2(z)\,\partial_z \ln\sqrt{g} - f_2(z)\partial_z \ln s_0^2 = \tfrac{1}{2}\widetilde{\mathcal{H}} f_1(\bar z)\, s_0^2\sqrt{g}^{\frac{1+n}{2}}\,.$$

Supersymmetric backgrounds: 1 supercharge ζ

$$\deg(\sqrt{K}^{-1\pm n}) = 1 - g \mp m.$$

$g > 1$

- $n = +1$: $(\zeta_-, \zeta_+) = (0, 1)$, $A = +\frac{1}{2}\omega$, $\mathcal{H} = 0$, $\forall \widetilde{\mathcal{H}}$ $\rightarrow$ A-twist
- $n = -1$: $(\zeta_-, \zeta_+) = (1, 0)$, $A = -\frac{1}{2}\omega$, $\widetilde{\mathcal{H}} = 0$, $\forall \mathcal{H}$ $\rightarrow$ $\overline{A}$-twist

$g = 1$: T^2

- Trivial bundles: constant f_1, f_2.

$g = 0$: S^2

- $n = +1$: A-twist, $U(1)$-equivariant A-twist
- $n = -1$: $\overline{A}$-twist, $U(1)$-equivariant $\overline{A}$-twist
- $n = 0$: $\overline{f_1(\overline{z})}, f_2(z) \in \mathcal{O}(1)$. One zero each.
  Can have single supercharge.

Supersymmetric backgrounds: more supercharges

Integrability conditions: ζ

$$2\zeta_- F_{1\bar{1}} + \tfrac{i}{4}\zeta_-(R - 2\mathcal{H}\widetilde{\mathcal{H}}) - i\zeta_+\partial_1\mathcal{H} = 0$$
$$2\zeta_+ F_{1\bar{1}} - \tfrac{i}{4}\zeta_+(R - 2\mathcal{H}\widetilde{\mathcal{H}}) + i\zeta_-\partial_{\bar{1}}\widetilde{\mathcal{H}} = 0$$

Integrability conditions: $\widetilde{\zeta}$

$$-2\widetilde{\zeta}_- F_{1\bar{1}} + \tfrac{i}{4}\widetilde{\zeta}_-(R - 2\mathcal{H}\widetilde{\mathcal{H}}) - i\widetilde{\zeta}_+\partial_1\widetilde{\mathcal{H}} = 0$$
$$-2\widetilde{\zeta}_+ F_{1\bar{1}} - \tfrac{i}{4}\widetilde{\zeta}_+(R - 2\mathcal{H}\widetilde{\mathcal{H}}) + i\widetilde{\zeta}_-\partial_{\bar{1}}\mathcal{H} = 0$$

$F = dA$, $\quad R$ Ricci scalar ($= -2/L^2$ on round S^2_L of radius L).

2+2 supercharges

$$F = dA = 0\,, \quad R = 2\mathcal{H}\widetilde{\mathcal{H}}\,, \quad \partial_\mu\mathcal{H} = 0\,, \quad \partial_\mu\widetilde{\mathcal{H}} = 0$$

- Flat T^2
- Round S^2_L with $\mathcal{H}\widetilde{\mathcal{H}} = -1/L^2$: [Benini, SC 2012; Doroud, Gomis, Le Floch, Lee 2012] and constant complexified $U(1)_{R_A}$ rotations thereof.

Supersymmetric backgrounds: more supercharges

2+0 supercharges

$$F = dA = 0\,, \quad R = 2\mathcal{H}\widetilde{\mathcal{H}}\,, \quad \partial_z\mathcal{H} = 0\,, \quad \partial_{\bar{z}}\widetilde{\mathcal{H}} = 0$$

1+1 supercharges: Alternatives

- $\exists$ globally defined Killing vector $K = \zeta\gamma^\mu\widetilde{\zeta}\,\partial_\mu$:
  - T^2
  - $U(1)$-isometric squashed S^2:

    $n = \pm 1$: ($U(1)$-equivariant) $A/\overline{A}$-twist
    $n = 0$: $f_1 = 1$, $f_2 = iz$ (and conjugate), generalizing [Gomis, Lee 2012].
- No isometries: $A/\overline{A}$-twist (possibly with $\widetilde{\mathcal{H}}$ or $\mathcal{H} \neq 0$).

Outlook

Outlook

- We have defined 2d $\mathcal{N} = (2,2)$ field theories with $U(1)_{R_V}$ symmetry in curved space.
- We have classified supersymmetric backgrounds.
- Subclasses of these backgrounds can uplift to 3d or 4d.
- We use these results to study Q-cohomology: whether and how the partition function and correlators depend on various deformations.
- We hope that our study will lead to new exact results in superstring theory, geometry and 4d gauge theories.