Volume Calculation for Domain-wall Moduli Spaces and Localization

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A. Miyake, KO and N. Sakai, Prog. Theor. Phys. 126 (2012) 637 [arXiv:1105.2087]
KO, N. Sakai and Y. Yoshida, PTEP 2013 (2013) 7 [arXiv:1303.4961]

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Introduction

Volume of moduli space of BPS solitons (instantons, monopoles, vortices, domain-walls, ...) gives

- Statistical mechanics of the BPS solitons
- Non-perturbative corrections in supersymmetric gauge theory
- Topological string amplitudes (topological invariants)

We evaluate the volume of the moduli space of the BPS solitons by the localization method (equivariant cohomology in mathematics)

Introduction

In particular, I consider today the volume of moduli space of the domain-walls

Why?

- Extension of the localization method to the case with boundaries (cf. 3d CS on S^3, 2d N=(2,2) on S^2, ... ⇒ without boundary)
  Boundary conditions are important!

- Important to solve BFSS like matrix quantum mechanics ((1+0)-dim)

Moduli space

- Moduli space: parameter space of solutions of the BPS solitons

$$\mathcal{M} = \frac{\cap_i \mu_i^{-1}(0)}{\{\text{gauge sym.}\}}$$

$$\mu_i: \ \text{moment maps}$$

e.g. $U(N)$ instantons on $\mathbf{C}^2 \ni (z,w)$

$$\mathcal{M}_k = \frac{\mu_r^{-1}(0) \cap \mu_c^{-1}(0) \cap \mu_c^{\dagger -1}(0)}{U(N)}$$

$$\boxed{\begin{aligned} \mu_r &= F_{z\bar{z}} + F_{w\bar{w}} \\ \mu_c &= F_{zw} \end{aligned}}$$

self-dual equation

$$k = \frac{1}{8\pi^2} \int \mathrm{Tr}\, F \wedge F$$

Volume of the moduli space

In general,

$$\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} d^n x \sqrt{g}$$

$$g \,:\, \text{metric of } \mathcal{M}, \qquad n = \dim \mathcal{M}$$

Moduli space ⇒ metric : difficult

Roughly speaking, we can formulate the volume as a "path integral" of the supersymmetric gauge theory

$$\mathrm{Vol}(\mathcal{M}) \sim \int \mathcal{D}(\text{fields}) \prod_i J_i \cdot \delta(\mu_i)$$

$$\sim \text{``}Z_{\mathrm{SYM}}\text{''}$$

Jacobians

BPS domain-walls

We consider the BPS domain-wall system:

$$\mu_r \equiv \mathcal{D}_y \Sigma - \frac{g^2}{2}\left(c\mathbf{1}_{N_c} - HH^\dagger\right) = 0 \qquad \text{D-term}$$

$$\left.\begin{aligned} \mu_c &\equiv \mathcal{D}_y H + \Sigma H - HM = 0 \\ \mu_c^\dagger &\equiv \mathcal{D}_y H^\dagger + H^\dagger \Sigma - MH^\dagger = 0 \end{aligned}\right) \quad \text{F-term}$$

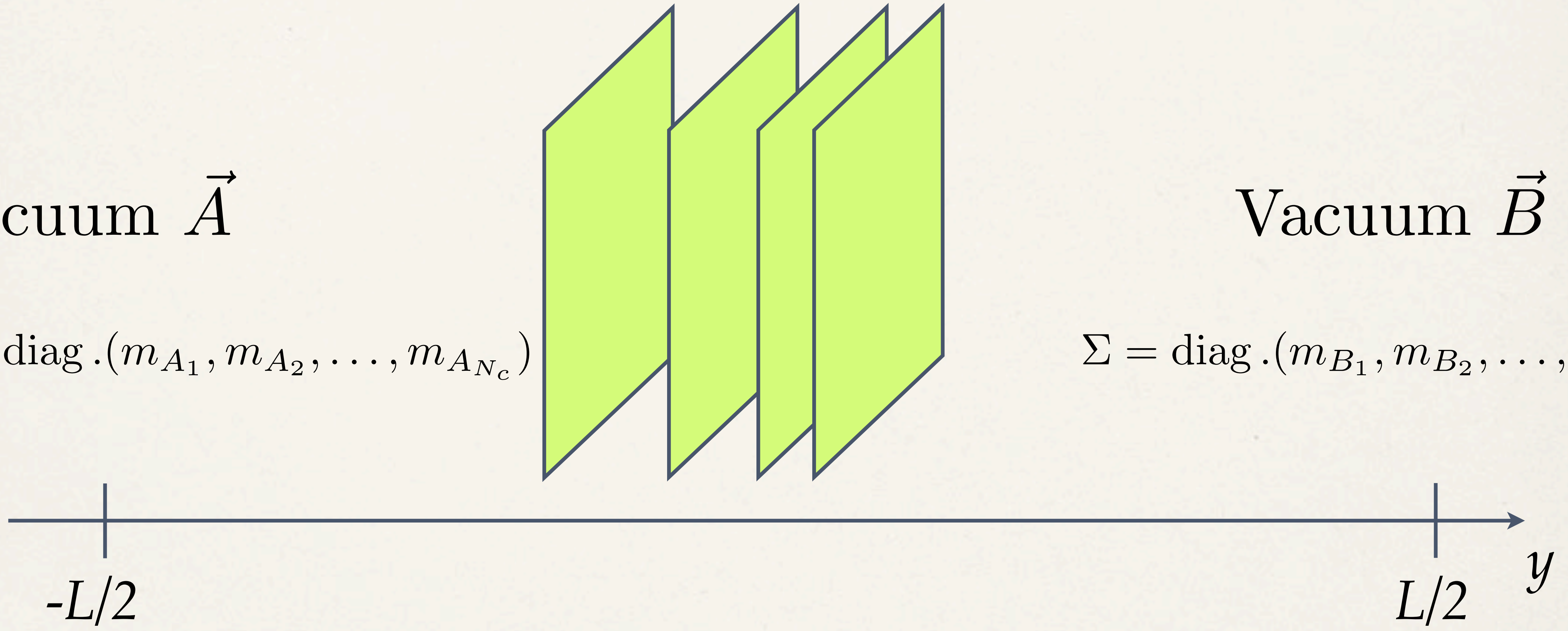
	G=$U(N_c)$	global $U(N_f)$
Σ	Adj.	1
H	1	N_f
$H^\dagger$	1	$\overline{N_f}$

g: gauge coupling, c: FI parameter

$$M = \mathrm{diag}\,.(m_1, m_2, \ldots, m_{N_f})$$

Boundary condition

domain-walls



Vacuum $\vec{A}$

$\Sigma = \mathrm{diag}\,.(m_{A_1}, m_{A_2}, \ldots, m_{A_{N_c}})$

Vacuum $\vec{B}$

$\Sigma = \mathrm{diag}\,.(m_{B_1}, m_{B_2}, \ldots, m_{B_{N_c}})$

Moduli space: $\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}} = \dfrac{\mu_r^{-1}(0)\cap\mu_c^{-1}(0)\cap\mu_c^{\dagger\,-1}(0)}{U(N_c)}$

BRST (super)symmetry

We introduce a fermionic symmetry (a part of SUSY)

$$
\begin{array}{ll}
QA_y = \lambda_y, & Q\lambda_y = -\mathcal{D}_y\Phi, \\
Q\Sigma = \xi, & Q\xi = i[\Phi, \Sigma] \\
Q\Phi = 0 & \\
QH = \psi, & Q\psi = i\Phi H \\
QY_c = i\Phi\chi_c, & Q\chi_c = Y_c
\end{array}
$$

$$Q^2 = -\delta_{\text{gauge}}(\Phi)$$

⬇

Equivariant cohomology

(nilpotent on gauge invariant operators)

Localization

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) &= \int \mathcal{D}(\text{fields})\, e^{-S_0 - tQ\Xi} \\ &= \left\langle e^{-S_0} \right\rangle_{\mathrm{SYM}}\end{aligned}$$

S_{SYM}

where

Lagrange multiplier

$$S_0 = i\beta \int_{-L/2}^{L/2} dy\, \mathrm{Tr}\left[\Phi \mu_r\right] + (\text{fermions}) \quad \underset{\text{impose}}{\Rightarrow} \quad \mu_r = 0$$

- Independent of t
- 1-loop exact
- localized at saddle (fixed) point

Contour integral

Fixed point (with a gauge fixing):

- Q(fields)=0 & $\mu_c=\mu_c^\dagger=0$
- $G=U(N_c) \to U(1)^{N_c}$
- Φ: const. on $y \Rightarrow \Phi=\mathrm{diag.}(\phi_1, \phi_2, \ldots, \phi_{N_c})$

1-loop det.

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \sum_{\sigma\in\mathfrak{S}_{N_c}} \prod_{a=1}^{N_c} \int_{-\infty}^{\infty} \frac{d\phi_a}{2\pi} \frac{(-1)^{|\sigma|}}{(i\phi_a)^{\mathrm{ind}\,P_a}} e^{i\beta\phi_a\left\{\hat{L}-(m_{B_{\sigma(a)}}-m_{A_a})\right\}}$$

S_0 at fixed point

where $\hat{L} \equiv \frac{g^2 c}{2} L$

$\mu_c=\mu_c^\dagger=0$

$$P_a H \equiv \mathcal{D}_y H + \Sigma H - H M$$

$$\mathrm{ind}\, P_a \Rightarrow \#\text{ of zero modes of } H = (\#\text{ of domain-walls}) + 1$$

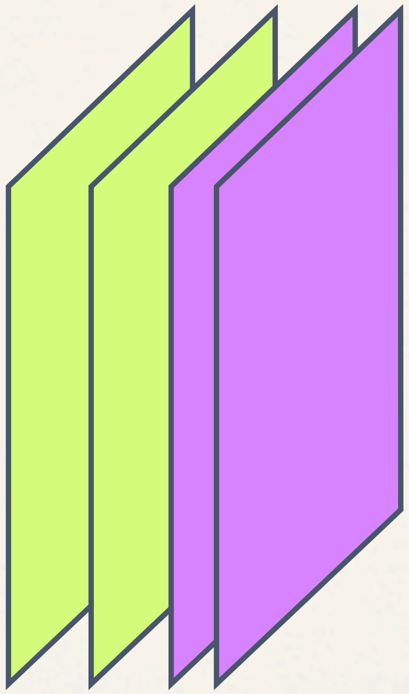
Example

Let us consider the case that N_c=2 and N_f=4 $(m_1 < m_2 < m_3 < m_4)$

The boundary condition:

$$\Sigma(-L/2) = \mathrm{diag}\,.(m_1, m_2)$$

$$\vec{A} = (1, 2)$$



$$\Sigma(L/2) = \mathrm{diag}\,.(m_3, m_4)$$

$$\vec{B} = (3, 4)$$

Total # of domain-walls = 4

Example

Using a residue formula:

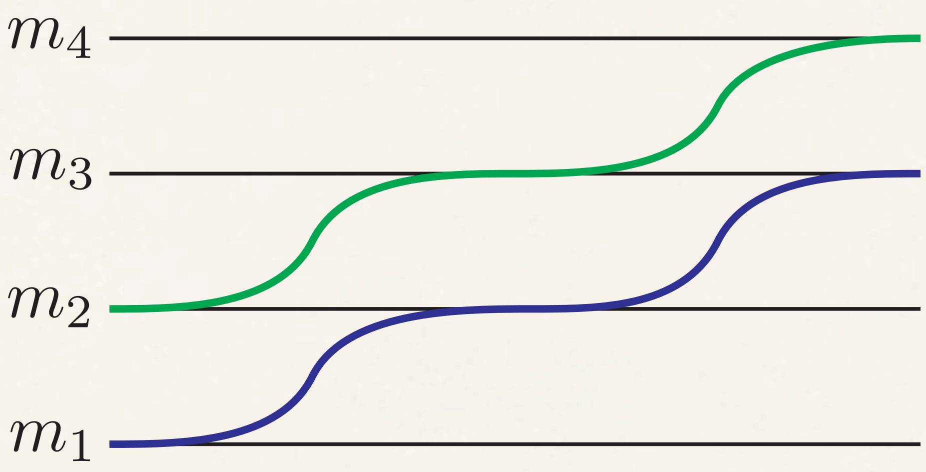
$$\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\phi}{2\pi i}\frac{1}{\phi^{n+1}}e^{i\phi B} = \begin{cases} \frac{1}{n!}(iB)^n & \text{if } B \geq 0 \text{ and } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

we find

$$\begin{aligned} \text{Vol}\left(\mathcal{M}^{2,4}_{(1,2)\to(3,4)}\right) &= \int \frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\frac{e^{i\beta\phi_1(\hat{L}-(m_3-m_1))}}{(i\phi_1)^3}\frac{e^{i\beta\phi_2(\hat{L}-(m_4-m_2))}}{(i\phi_2)^3} \\ &\quad - \int \frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\frac{e^{i\beta\phi_1(\hat{L}-(m_4-m_1))}}{(i\phi_1)^4}\frac{e^{i\beta\phi_2(\hat{L}-(m_3-m_2))}}{(i\phi_2)^2} \\ &= \beta^4\left\{\frac{1}{4}(\hat{L}-(m_3-m_1))^2(\hat{L}-(m_4-m_2))^2\right. \\ &\quad \left. - \frac{1}{6}(\hat{L}-(m_4-m_1))^3(\hat{L}-(m_3-m_2))\right\} \end{aligned}$$

Example

$$\mathrm{Vol}\left(\mathcal{M}^{2,4}_{(1,2)\to(3,4)}\right) =$$



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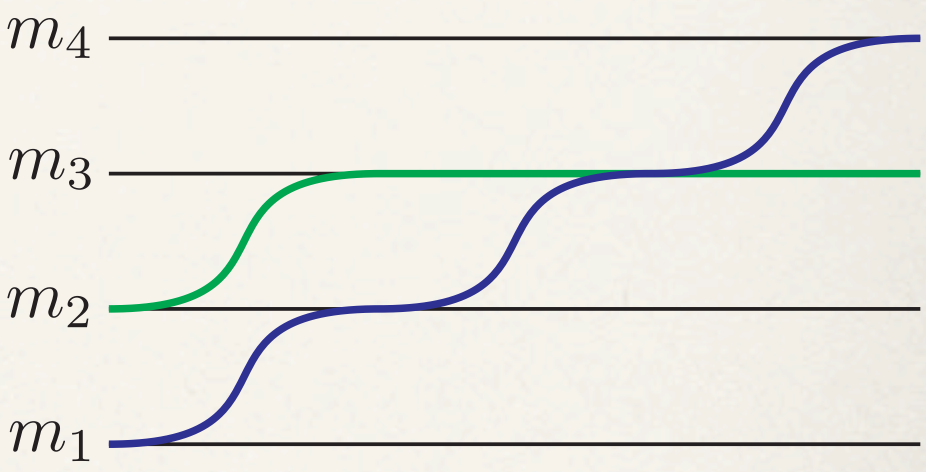


$$= \det \begin{array}{ccc} & m_3 & m_4 \\ m_1 & \frac{1}{2!}(\hat{L} - (m_3 - m_1))^2 & \frac{1}{3!}(\hat{L} - (m_4 - m_1))^3 \\ m_2 & \hat{L} - (m_3 - m_2) & \frac{1}{2!}(\hat{L} - (m_4 - m_2))^2 \end{array}$$

Transition matrix

Seiberg like duality

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) \neq \mathrm{Vol}\left(\mathcal{M}^{N_f-N_c,N_f}_{\bar{\vec{B}}\to\bar{\vec{A}}}\right)$$

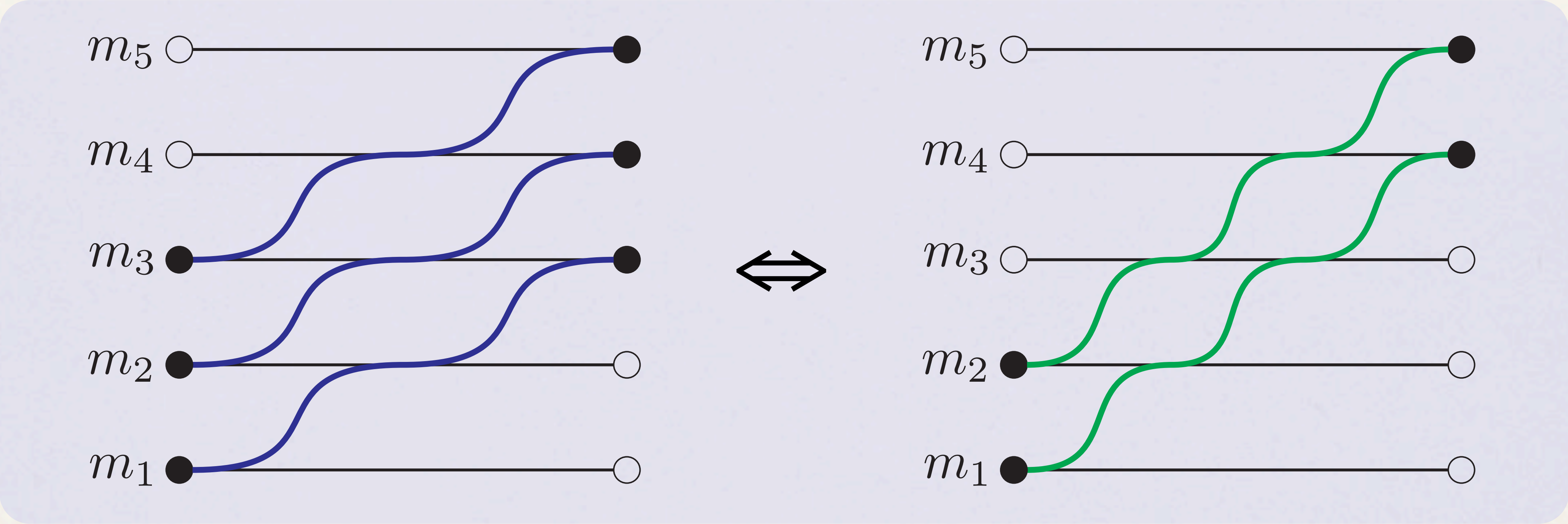
$$\hat{L}\to\infty$$

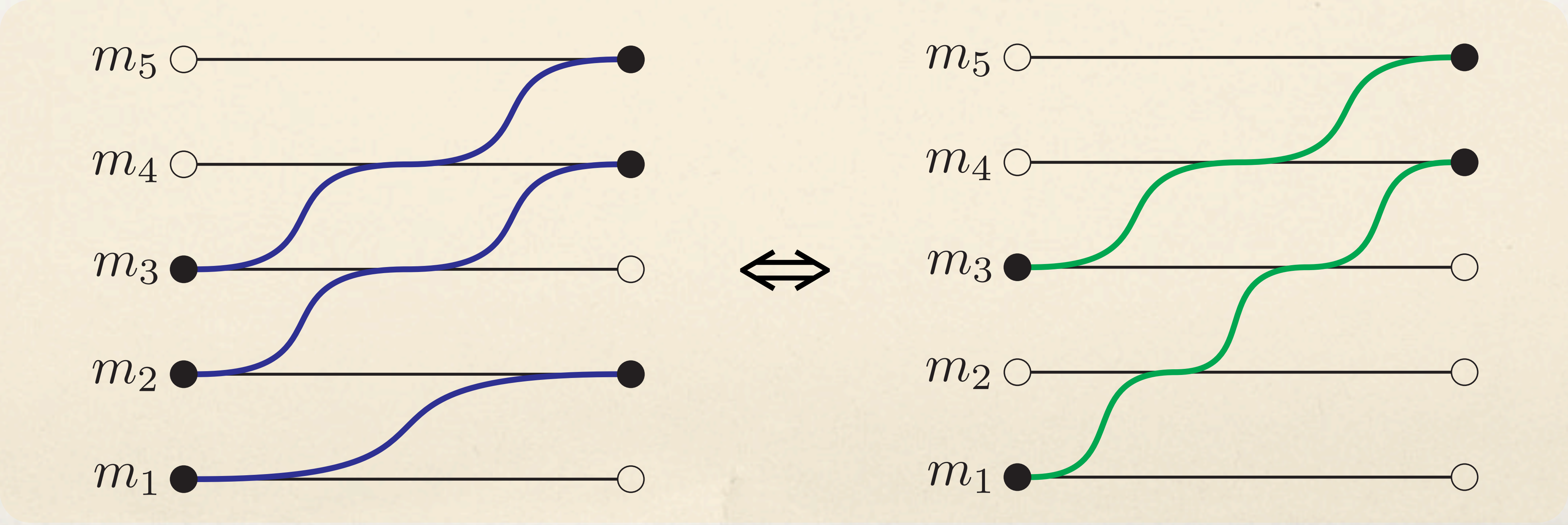
$$(g\to\infty)$$

$$\frac{\prod_{j=1}^{N_c}(j-1)!\times\prod_{k=1}^{\tilde{N}_c}(k-1)!}{\prod_{i=1}^{N_f}(i-1)!}\left(\beta\hat{L}\right)^{N_c(N_f-N_c)}$$

Volume of the Grassmaniann: $G_{N_c,N_f}\equiv\dfrac{U(N_f)}{U(N_c)\times U(\tilde{N}_c)}$

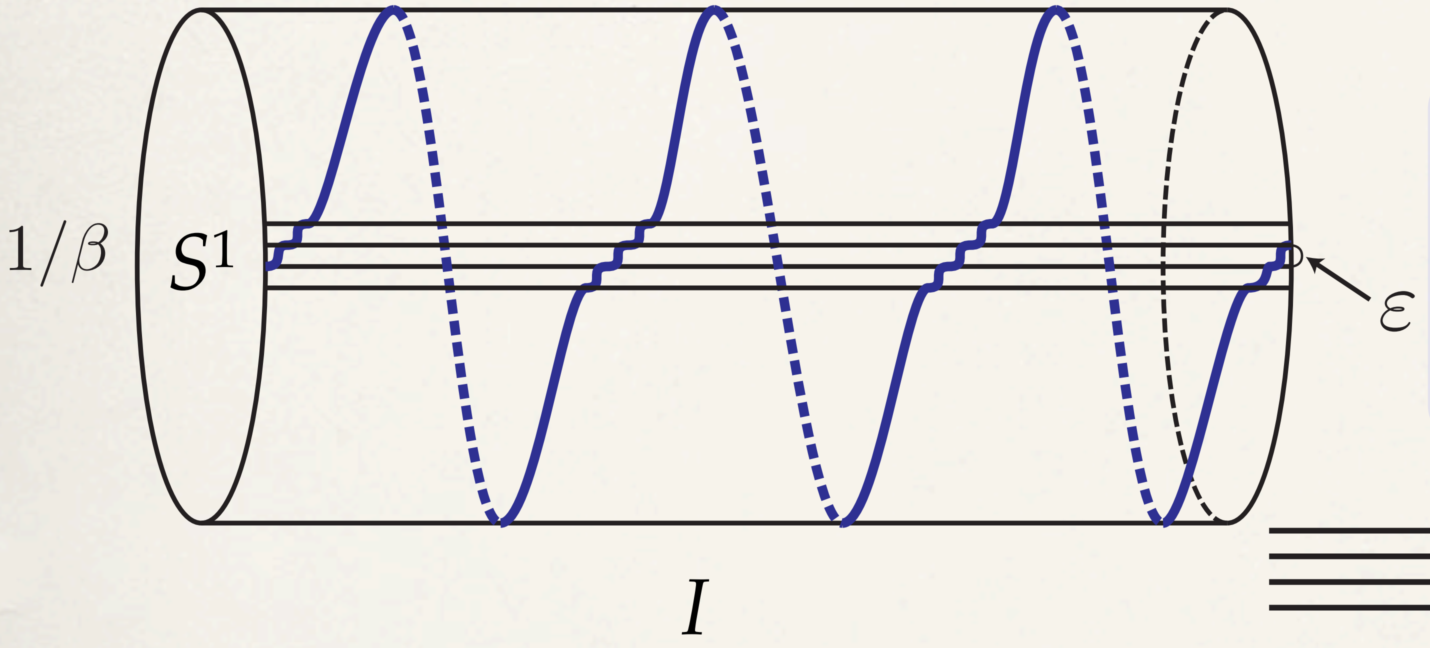
Seiberg like duality



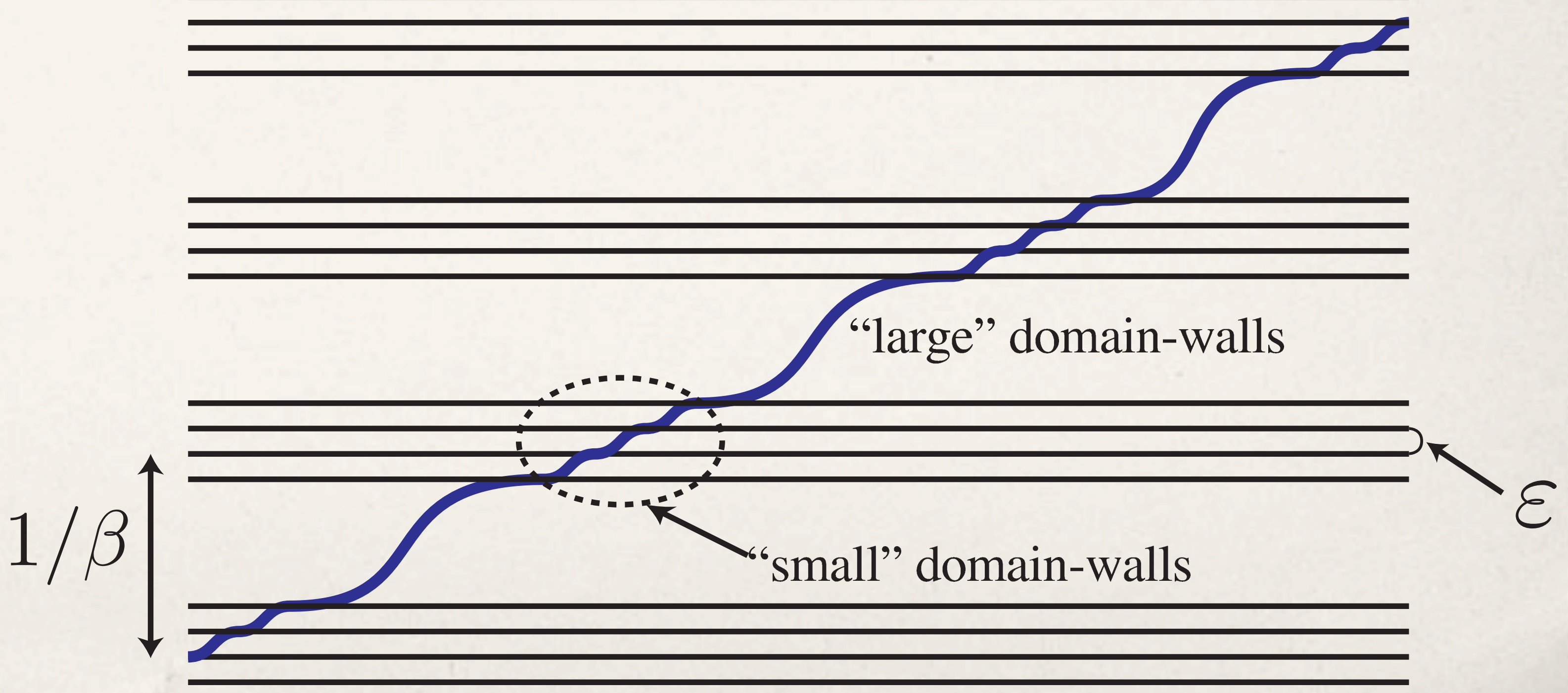


T-duality

We consider domain-walls on a cylinder $S^1 \times I$



⇔ vortices on a cylinder $S^1 \times I$
(k=3 with infinitesimal masses
and holonomy)



T-duality

In particular, the N_c=N_f=2 case (local vortex)

$$\mathrm{Vol}\left(\mathcal{M}_0^{2,2}(S^1\times I)\right)=1$$

$$\mathrm{Vol}\left(\mathcal{M}_1^{2,2}(S^1\times I)\right)=\hat{\mathcal{A}}-1+\hat{\varepsilon}-\hat{\varepsilon}^2$$

$$\mathrm{Vol}\left(\mathcal{M}_2^{2,2}(S^1\times I)\right)=\frac{1}{2}\hat{\mathcal{A}}^2-\left(\frac{5}{3}-\hat{\varepsilon}+\hat{\varepsilon}^2\right)\hat{\mathcal{A}}+\frac{17}{12}-\frac{5}{3}\hat{\varepsilon}+2\hat{\varepsilon}^2-\frac{2}{3}\hat{\varepsilon}^3+\frac{1}{3}\hat{\varepsilon}^4$$

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}_3^{2,2}(S^1\times I)\right)=&\frac{1}{6}\hat{\mathcal{A}}^3-\frac{1}{2}\left(\frac{7}{3}-\hat{\varepsilon}+\hat{\varepsilon}^2\right)\hat{\mathcal{A}}^2\\&+\left(\frac{331}{120}-\frac{7}{3}\hat{\varepsilon}+\frac{8}{3}\hat{\varepsilon}^2-\frac{2}{3}\hat{\varepsilon}^3+\frac{1}{3}\hat{\varepsilon}^4\right)\hat{\mathcal{A}}\\&-\frac{793}{360}+\frac{331}{120}\hat{\varepsilon}-\frac{85}{24}\hat{\varepsilon}^2+\frac{29}{18}\hat{\varepsilon}^3-\frac{11}{12}\hat{\varepsilon}^4+\frac{2}{15}\hat{\varepsilon}^5-\frac{2}{45}\hat{\varepsilon}^6\end{aligned}$$

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}_4^{2,2}(S^1\times I)\right)=&\frac{1}{24}\hat{\mathcal{A}}^4-\frac{1}{6}\left(3-\hat{\varepsilon}+\hat{\varepsilon}^2\right)\hat{\mathcal{A}}^3\\&+\frac{1}{2}\left(\frac{409}{90}-3\hat{\varepsilon}+\frac{10}{3}\hat{\varepsilon}^2-\frac{2}{3}\hat{\varepsilon}^3+\frac{1}{3}\hat{\varepsilon}^4\right)\hat{\mathcal{A}}^2\\&-\left(\frac{292}{63}-\frac{409}{90}\hat{\varepsilon}+\frac{111}{20}\hat{\varepsilon}^2-\frac{37}{18}\hat{\varepsilon}^3+\frac{41}{36}\hat{\varepsilon}^4-\frac{2}{15}\hat{\varepsilon}^5+\frac{2}{45}\hat{\varepsilon}^6\right)\hat{\mathcal{A}}\\&+\frac{18047}{5040}-\frac{292}{63}\hat{\varepsilon}+\frac{37}{6}\hat{\varepsilon}^2-\frac{16}{5}\hat{\varepsilon}^3+\frac{35}{18}\hat{\varepsilon}^4-\frac{19}{45}\hat{\varepsilon}^5+\frac{7}{45}\hat{\varepsilon}^6-\frac{4}{315}\hat{\varepsilon}^7+\frac{1}{315}\hat{\varepsilon}^8\end{aligned}$$

$\hat{\mathcal{A}}:$ area of $S^1\times I$

$$\mathrm{Vol}\left(\mathcal{M}_k^{N,N}\right)\sim\frac{\hat{\mathcal{A}}^k}{k!}$$

In the $\hat{\varepsilon}\to 0$ limit, the volume agrees with that of the local vortices on S^2

Conclusion and Discussion

Results:

- We exactly evaluate the volume of the moduli space of the domain-walls via the localization method
- The volume is given by the simple contour integral
- We find the dualities between the moduli spaces

Problems:

- Relation to integrable systems (spin chain, etc.)
- Relation to string/M theory (some topological invariants)