# Accidental Supersymmetry and the Renormalization of Codimension-2 Branes


Matt Williams


August 26, 2013

Based on hep-th/1210.3753, hep-th/1210.5405
with C.P. Burgess, L. van Nierop, S. Parameswaran, and A. Salvio.

# Outline

# Take-home Message

1. The Salam-Sezgin solution in 6D SUGRA can be deformed by 4D sources in a way that preserves SUSY
2. Surprising result: fields can dynamically adjust to preserve SUSY for *arbitrary 4D source tensions, charges*
3. Such a construction has desirable phenomenological consequences for the naturalness of dark energy

# 6D Supergravity: Salam-Sezgin Solution

Action for non-trivial bosonic fields:

$$S = -\int \mathrm{d}^6 X \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \partial_M \phi \, \partial^M \phi \right) + \frac{e^{-\phi}}{4g_R^2} F_{MN} F^{MN} + \frac{2g_R^2}{\kappa^4} \, e^{\phi} \right]$$

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Background:

$$\mathrm{d}s^2 = \eta_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2)$$
$$F_{\mu N} = 0 \,, \quad F_{mn} = f \, \varepsilon_{mn} \,, \quad \phi = \text{const.}$$

where $r \to$ radius of the extra-dimensional sphere
and $\varepsilon_{mn} \to$ extra-dimensional volume form.

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Salam-Sezgin solution: (Phys. Lett. **B147** (1984) 47)

$$F_{\theta\varphi} = \pm \frac{\varepsilon_{\theta\varphi}}{2r^2} = \pm \frac{\sin\theta}{2} \,, \quad r^2 e^{\phi} = \frac{\kappa^2}{4g_R^2} \,.$$

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$$(b = \pm 1 \;\leftarrow \text{N/S pole})$$

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$$\left(\omega_{\varphi 45} = \alpha\cos\theta - b\,, \quad A_\varphi = \mp\frac{\alpha}{2}(\cos\theta - b) + b\,\Phi_b\right)$$

$$\to \Phi_b = \pm\frac{1}{2}(1-\alpha)$$

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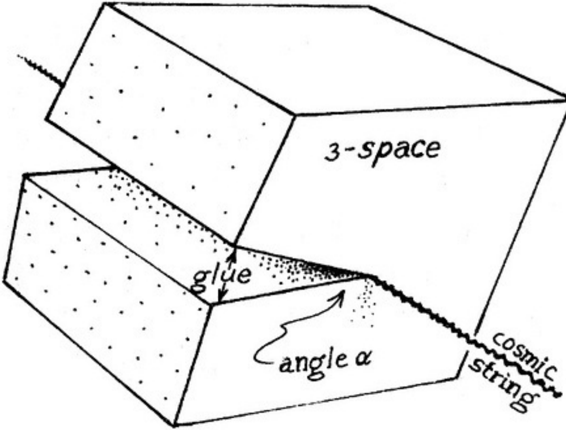
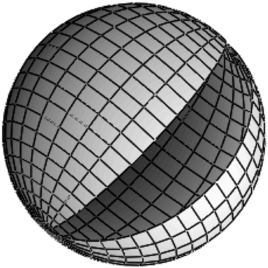
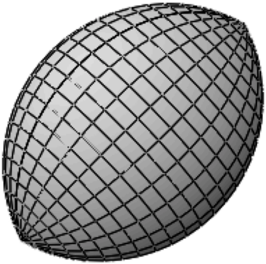
# Motivation: Analogy with Cosmic Strings




Picture credit: R. Penrose, *The Road to Reality*

# Rugby-Ball Background





$$S_b \supset -\int \mathrm{d}^4x\sqrt{-g_4}\,T_b\,, \quad b = \pm 1\,.$$

Background Metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + r^2\left(\mathrm{d}\theta^2 + \alpha^2 \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

where

$$\alpha = 1 - \frac{\kappa^2 T_+}{2\pi} = 1 - \frac{\kappa^2 T_-}{2\pi}\,.$$

Deficit angle: $\delta = 2\pi(1-\alpha)$.

# Brane Magnetic Charge

$$S_b \supset \frac{\mathcal{A}_b}{2g_R^2}\int {}^*F = \int \mathrm{d}^4x\sqrt{-g_4}\,\frac{\mathcal{A}_b}{2g_R^2}\,\varepsilon^{mn}F_{mn}\,.$$
$$\left(\mathcal{L}_{gf} = -\frac{e^{-\phi}}{4g_R^2}F_{MN}F^{MN}\right)$$

Alters gauge field boundary conditions ($b = \pm 1 \leftarrow$ N/S pole):

$$A_\varphi\Big|_{\cos\theta = b} = b\,\frac{e^{\phi}\mathcal{A}_b}{2\pi} := b\,\Phi_b\,.$$

# Flux Quantization

Non-local condition:

$$N = \sum_b \Phi_b + \frac{1}{2\pi} \int \mathrm{d}^2 y \, F_{\theta\varphi} \,.$$

Gauge field strength:

$$F_{\theta\varphi} = \pm \frac{\varepsilon_{\theta\varphi}}{2r^2} = \pm \frac{\alpha \sin\theta}{2} \,;$$

$$\to \sum_b \Phi_b = \pm(1-\alpha) \,.$$

Identical branes:

$$\Phi_b = \pm \frac{1}{2}(1-\alpha) \quad \text{or} \quad e^{\phi_b} \mathcal{A}_b = \pm \frac{\kappa^2 T_b}{2} \,.$$

## Application: 4D Vacuum Energy at 1-loop

Taking $\mathcal{A}_+ \neq \mathcal{A}_-$ provides a framework to check the technical naturalness of the classical result:

$$\rho_V \Big|_{\text{class.}} = \frac{1}{2} \frac{\partial \mathcal{L}_b}{\partial \phi} = 0 \,.$$

( L. van Nierop and C.P. Burgess in hep-th/1108.0345.)

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Since $\Sigma_{1L} = -\int d^4x\,V_{1L} = \frac{i}{2}\ln\mathrm{Det}(-D^M D_M + m^2)$,

$$\rightarrow \quad V_{1L} = \frac{1}{2}\,\mu^{4-d}\sum_{jn}\int\frac{\mathrm{d}^d k_E}{(2\pi)^d}\,\ln\left(\frac{k_E^2 + m_{jn}^2 + m^2}{\mu^2}\right).$$

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Heat kernel expansion:

$$V_{1L} = -\frac{\mu^{4-d}}{2(4\pi r^2)^{d/2}} \int_0^\infty \frac{\mathrm{d}t}{t^{1+d/2}}\, e^{-t(mr)^2} \left( \sum_{j,n} e^{-t\,(m_{jn} r)^2} \right) .$$

# Technical Details

- Obtain KK spectrum for a scalar field (or spin-$\frac{1}{2}$, etc.)
- Perform the sum over KK modes
- Extract the UV divergences by taking $\lim_{d \to 4} V_{1L}$

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Final result:

$$V_{1L} = \frac{\mathcal{C}}{(4\pi)^2}\left[\frac{1}{4-d} + \ln\left(\frac{\mu}{m}\right)\right] + \cdots$$

where

$$\mathcal{C} = \frac{s_{-1}}{6}\, m^6 r^2 - \frac{s_0}{2}\, m^4 + s_1\, \frac{m^2}{r^2} - \frac{s_2}{r^4}\,.$$

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$$s_i = \alpha s_i^{\rm sph} + \sum_b \delta s_{i(b)}(\alpha, \Phi_b)\,.$$

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- Identify the requisite bulk, brane counterterms
- Obtain beta functions for renormalized quantities

# Result: Massive Multiplet

Back-reaction: $\rho_{V(1L)} = -\frac{1}{2}\,\frac{\partial V_{R(1L)}}{\partial \phi}$ ;

- Massive multiplet: $(\eta = |\Delta\Phi|/(2\alpha),\ m^2(\phi) = M^2 e^{\phi})$

$$\rho_{V(1L)} = \frac{\eta}{(4\pi r^2)^2}\left[\left(\frac{\kappa M}{2\,g_R}\right)^4 - \left(\frac{1}{3} + \frac{(1-\alpha^2)}{2\alpha}\eta - \frac{\eta^2}{3}\right)\left(\frac{\kappa M}{2\,g_R}\right)^2\right]\ln\left(\frac{M_{\rm s}}{M}\right) .$$

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Technically natural? ✓

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Thank you for your attention!

For more information:


- M. Williams, C. P. Burgess, L. van Nierop and A. Salvio, "Running with Rugby Balls: Bulk Renormalization of Codimension-2 Branes," JHEP **1301**, 102 (2013) [arXiv:1210.3753 [hep-th]]
- C. P. Burgess, L. van Nierop, S. Parameswaran, A. Salvio and M. Williams, "Accidental SUSY: Enhanced Bulk Supersymmetry from Brane Back-reaction," JHEP **1302**, 120 (2013) [arXiv:1210.5405 [hep-th]]

# Example — Scalar Loop Calc. (cont'd)

- Solve for scalar's KK spectrum, $m^2_{jn}(\alpha, N, \Phi_b)$:

$$m^2_{jn} = \frac{1}{r^2}\left[(j+y+z)(j+y+z+1) - \frac{\mathcal{N}^2}{4}\right]$$

where $y = \frac{1}{2\alpha}|n - \Phi_+|$, $z = \frac{1}{2\alpha}|n - N + \Phi_-|$, and $\mathcal{N} = \frac{N-\Phi}{\alpha}$.

(When $1 - \alpha = N = \Phi_b = 0$, $m^2_{jn} = \frac{\ell(\ell+1)}{r^2}$ with $\ell := j + |n|$.)

- Sum over KK spectrum using E.–M., Poisson resummation:

$$\sum_{j,n} e^{-t\,(m_{jn}r)^2} = \frac{s_{-1}}{t} + s_0 + s_1 t + s_2 t^2 + \mathcal{O}(t^2)\,;$$

- Take limit $d \to 4$:

$$V_{1L} = \frac{\mathcal{C}}{(4\pi)^2}\left[\frac{1}{4-d} + \ln\left(\frac{\mu}{m}\right)\right] + \mathcal{V}_f\,,$$

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# “Bulk Renorm. is Independent of BC’s”

Sphere, no BLF’s ($\alpha = 1, \Phi_b = 0$): (Kantowski & Milton, 1987)

$$s_{-1}^{\text{sph}} = 1\,, \quad s_0^{\text{sph}} = \frac{1}{3}\,, \quad s_1^{\text{sph}} = \frac{1}{15} - \frac{N^2}{24}\,, \quad s_2^{\text{sph}} = \frac{4}{315} - \frac{N^2}{40}\,.$$

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Required bulk counterterms:

$$\mathcal{L}_{1L\rm ct} = -\Big[U + \frac{1}{2\kappa^2}\,R + \frac{\zeta_{R^2}}{\kappa}\,\overline{R}^2 + \frac{1}{4g^2}\,F_{MN}F^{MN} + \zeta_{R^3}\,\overline{R}^3 + \frac{\kappa\zeta_{AR}}{8g^2}\,R\,F_{MN}F^{MN}\Big]$$

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Running:

$$\mu\,\frac{\partial U}{\partial\mu} = -\frac{m^6}{6(4\pi)^3}\,, \quad \mu\,\frac{\partial}{\partial\mu}\left(\frac{1}{\kappa^2}\right) = -\frac{m^4}{6(4\pi)^3}\,, \quad \mu\,\frac{\partial}{\partial\mu}\left(\frac{\zeta_{R^2}}{\kappa}\right) = -\frac{m^2}{60(4\pi)^3}\,,$$

$$\mu\,\frac{\partial}{\partial\mu}\left(\frac{1}{g^2}\right) = \frac{2\,q^2m^2}{3(4\pi)^3}\,, \quad \mu\,\frac{\partial\zeta_{R^3}}{\partial\mu} = -\frac{1}{630(4\pi)^3}\,, \quad \mu\,\frac{\partial}{\partial\mu}\left(\frac{\kappa\zeta_{AR}}{g^2}\right) = \frac{2\,q^2}{5(4\pi)^3}\,.$$

# The General Case

$$s^{\mathrm{s}}_{-1} = \alpha\,,$$

$$s^{\mathrm{s}}_{0}(\omega, N, \Phi_b) = \alpha\left[\frac{1}{6} + \frac{\omega^2}{6}(1-3F)\right]\,,$$

$$s^{\mathrm{s}}_{1}(\omega, N, \Phi_b) = \alpha\left[\frac{1}{180} - \frac{\mathcal{N}^2}{24} + \frac{\omega^2}{18}(1-3F) - \frac{\omega^3\mathcal{N}}{12}\sum_b \Phi_b\, G(|\Phi_b|) + \frac{\omega^4}{180}(1-15F^{(2)})\right],$$

$$s^{\mathrm{s}}_{2}(\omega, N, \Phi_b) = \alpha\left[-\frac{1}{504} - \frac{11\,\mathcal{N}^2}{720} + \left(\frac{1}{90} - \frac{\mathcal{N}^2}{144}\right)(1-3F)\omega^2 - \frac{\omega^3\mathcal{N}}{24}\sum_b \Phi_b\, G(|\Phi_b|) + \frac{\omega^4(1-\mathcal{N}^2)}{360}(1-15F^{(2)}) - \frac{\omega^5\mathcal{N}}{120}\sum_b \Phi_b\, G(|\Phi_b|)(1+3F_b) + \left(\frac{1}{1260} - \frac{F^{(2)}}{120} - \frac{F^{(3)}}{60}\right)\omega^6\right].$$

with

$$\omega := \alpha^{-1}\,, \quad \mathcal{N} := \frac{N-\Phi}{\alpha}\,, \quad F_b := |\Phi_b|\,(1-|\Phi_b|)\,, \quad F^{(n)} := \sum_b F_b^n\,,$$

$$F := F^{(1)}\,, \quad G(|\Phi_b|) := (1-|\Phi_b|)(1-2|\Phi_b|)\,.$$

# “Brane Renorm. Depends Only on Itself”

- Bulk renormalization is independent of brane physics — contribution to $V_{1L}$ scales with $\alpha$ (volume factor);
- Brane renormalization vanishes in the limit $\alpha \to 1$, $\Phi_b \to 0$.

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$$s_i = \alpha s_i^{\text{sph}} + \sum_b \delta s_{i(b)}$$

Brane counterterms:

$$\mathcal{L}_{b,\text{ct}} = -\sqrt{-g_4}\left(T_{b,\text{ct}} - \frac{\mathcal{A}_{b,\text{ct}}}{2g_R^2}\,\epsilon \cdot F + \frac{\zeta_{R\,b}}{\kappa}\,R + \frac{\kappa\zeta_{A b}}{4g_R^2}\,F^2 + \frac{\kappa\zeta_{\tilde{A}R\,b}}{2g_R^2}\,R\,\epsilon \cdot F + \zeta_{R^2 b}\,\overline{R}^2\right) .$$

# “Brane Renorm. Depends Only on Itself”

- Bulk renormalization is independent of brane physics — contribution to $V_{1L}$ scales with $\alpha$ (volume factor);
- Brane renormalization vanishes in the limit $\alpha \to 1$, $\Phi_b \to 0$.

$$s_i = \alpha s_i^{\rm sph} + \sum_b \delta s_{i(b)}$$

Brane counterterms:

$$\mathcal{L}_{b,\rm ct} = -\sqrt{-g_4}\left(T_{b,\rm ct} - \frac{\mathcal{A}_{b,\rm ct}}{2g_R^2}\,\epsilon\cdot F + \frac{\zeta_{R\,b}}{\kappa}\,R + \frac{\kappa\zeta_{Ab}}{4g_R^2}\,F^2 + \frac{\kappa\zeta_{\tilde{A}R\,b}}{2g_R^2}\,R\,\epsilon\cdot F + \zeta_{R^2 b}\,\overline{R}^2\right).$$

Example: $(\omega := \alpha^{-1},\ \delta\omega := \omega - 1)$

$$\mu\,\frac{\partial T_b}{\partial\mu} = \frac{m^4}{2(4\pi)^2\omega}\left(\frac{\delta\omega}{6} + \frac{\delta\omega^2}{12} - \frac{\omega^2}{2}|\Phi_b|(1-|\Phi_b|)\right),$$

$$\mu\,\frac{\partial}{\partial\mu}\left(\frac{\mathcal{A}_b}{g_R^2}\right) = -\frac{\Phi_b\,\omega^2 m^2}{6(4\pi)^2}(1-|\Phi_b|)(1-2|\Phi_b|)\,.$$